# Sets of Marginals and Pearson-Correlation-based CHSH Inequalities for a Two-Qubit System

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ISIT 2021
Melbourne, Australia

# Outline

- Introduction
  - Probability Mass Functions (PMFs)
  - Quantum Mass Functions (QMFs)
  - Simple Quantum Mass Functions (SQMFs)
- Main Results

# Introduction

# PMFs

Consider a sequence $Y_1, \dots, Y_n$ of random variables with the joint PMF

$$P_{Y_1,\dots,Y_n}(y_1, \dots, y_n), \qquad y_1 \in \mathcal{Y}_1, \dots, y_n \in \mathcal{Y}_n.$$

In a typical scenario of interest, we might have observed

$$Y_1 = y_1, \dots, Y_{n-1} = y_{n-1}$$

and would like to estimate $Y_n$ based on these observations.

Usually, $P_{Y_1,\dots,Y_n}(y_1, \dots, y_n)$ does not have a "nice" factorization.

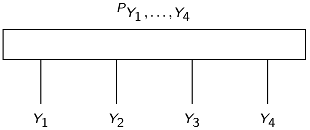
However, very often it is possible to find a function $p(\boldsymbol{x}, \boldsymbol{y})$ such that

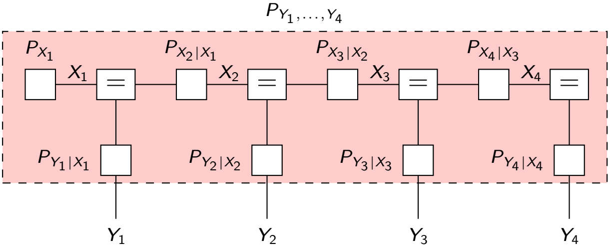
1. $p(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_{\geq 0}$ for all $\boldsymbol{x}, \boldsymbol{y}$;
2. $\sum_{\boldsymbol{x},\boldsymbol{y}} p(\boldsymbol{x}, \boldsymbol{y}) = 1$;
3. $\sum_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{y}) = P_{\boldsymbol{Y}}(\boldsymbol{y})$ for all $\boldsymbol{y}$;
4. $p(\boldsymbol{x}, \boldsymbol{y})$ has a "nice" factorization.

Note that $p(\boldsymbol{x}, \boldsymbol{y})$ represents a joint PMF over $\boldsymbol{x}$ and $\boldsymbol{y}$.

# PMFs

## Example (A Hidden Markov Chain)




After applying a closing-the-box (CTB) operation to the above factor graph, i.e., summing over the variables associated with the internal edges, we obtain $P_{Y_1,\ldots,Y_4}$.

# QMFs

Consider again a sequence $Y_1, \ldots, Y_n$ of random variables with the joint PMF

$$P_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n), \qquad y_1 \in \mathcal{Y}_1, \ldots, y_n \in \mathcal{Y}_n.$$

However, now we assume that these random variables represent the measurements obtained by running some quantum-mechanical experiment.

Again, a typical scenario of interest is that we would like to estimate $Y_n$ based on the observations

$$Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}.$$

# QMFs

In general, the PMF $P_{\boldsymbol{Y}}(\boldsymbol{y})$ does not have a "nice" factorization.

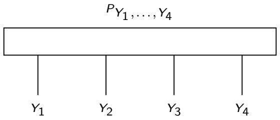
However, frequently it is possible to introduce suitable auxiliary variables $x_1, \ldots x_m, x'_1, \ldots, x'_m$ such that there is a function $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ satisfying

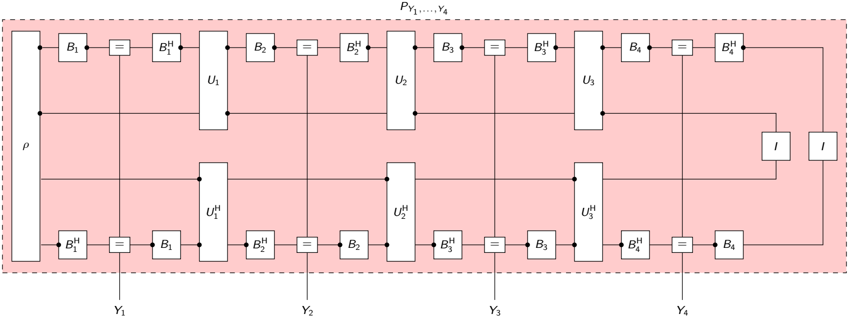
1. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) \in \mathbb{C}$ for all $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}$;
2. $\sum_{\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = 1$;
3. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ is a positive semi-definite (PSD) kernel in $(\boldsymbol{x}, \boldsymbol{x}')$ for every $\boldsymbol{y}$;
4. $\sum_{\boldsymbol{x}, \boldsymbol{x}'} q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = P_{\boldsymbol{Y}}(\boldsymbol{y})$;
5. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ has a "nice" factorization.

The function $q$ is called a quantum mass function (QMF) in [Loeliger and Vontobel, 2017].

# QMFs

## Example




After applying a CTB operation to the above factor graph, i.e., summing over the variables associated with the internal edges, we obtain $P_{Y_1,\ldots,Y_4}$.

# SQMFs

In [Loeliger and Vontobel, 2020], the authors also introduced simple quantum mass functions (SQMFs).

An SQMF $q(\boldsymbol{x}, \boldsymbol{x}')$ satisfies

1. $q(\boldsymbol{x}, \boldsymbol{x}') \in \mathbb{C}$ for all $\boldsymbol{x}, \boldsymbol{x}'$;
2. $\sum_{\boldsymbol{x}, \boldsymbol{x}'} q(\boldsymbol{x}, \boldsymbol{x}') = 1$;
3. $q(\boldsymbol{x}, \boldsymbol{x}')$ is a PSD kernel in $(\boldsymbol{x}, \boldsymbol{x}')$.

**Remark**

*Observations* $\boldsymbol{y}$ *in QMFs do not appear explicitly in SQMFs anymore. However, as we will see later, observations* $\boldsymbol{y}$ *emerge from SQMFs.*

# SQMFs

## Definition

For $\boldsymbol{x} = (x_1, \ldots, x_m)$, let $\mathcal{I} \subseteq \{1, \ldots, m\}$ and let $\mathcal{I}^{\mathrm{c}}$ be its complement. The variables $\boldsymbol{x}_{\mathcal{I}}$ are called jointly **classicable** if the marginalized SQMF

$$q_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}'_{\mathcal{I}}) := \sum_{\boldsymbol{x}_{\mathcal{I}^{\mathrm{c}}}, \boldsymbol{x}'_{\mathcal{I}^{\mathrm{c}}}} q(\boldsymbol{x}, \boldsymbol{x}')$$

is zero for all $(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}'_{\mathcal{I}})$ satisfying $\boldsymbol{x}_{\mathcal{I}} \neq \boldsymbol{x}'_{\mathcal{I}}$.

## Definition

If the variables $\boldsymbol{x}_{\mathcal{I}}$ are jointly classicable then

$$p(\boldsymbol{x}_{\mathcal{I}}) := q_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}_{\mathcal{I}}), \qquad \boldsymbol{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}},$$

represents a joint PMF over $\boldsymbol{x}_{\mathcal{I}}$.

## Definition

Let $\mathcal{K}$ be a collection of subsets $\mathcal{I}$ of $\{1, \ldots, m\}$ such that $\boldsymbol{x}_{\mathcal{I}}$ is classicable.

# SQMFs vs. QMFs vs. PMFs

## Remark

- *By defining* $p(\boldsymbol{x}_{\mathcal{I}}) := q_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}_{\mathcal{I}})$*, we can see that the observations* $\boldsymbol{y}$ *that were omitted when going from QMFs to SQMFs can "emerge" again.*
- *Typically, the set of marginals* $\{p(\boldsymbol{x}_{\mathcal{I}})\}_{\mathcal{I} \in \mathcal{K}}$ *is "incompatible", i.e., there is no PMF* $p(\boldsymbol{x})$ *such that* $p_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}})$ *is a marginal of* $p(\boldsymbol{x})$ *for all* $\mathcal{I} \in \mathcal{K}$*.*
- *Note that there is a strong connection of SQMFs to the so-called decoherence functional [Gell-Mann and Hartle, 1989, Dowker and Halliwell, 1992], and via this also to the consistent-histories approach to quantum mechanics [Griffiths, 2002]. However, the starting point of our investigations is quite different.*

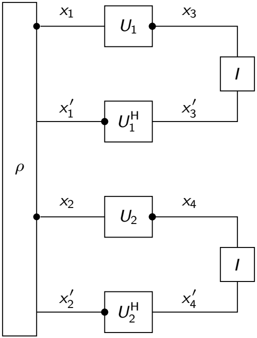
# SQMF for Hardy's Paradox

## Example

Consider the following quantum factor graph (Q-FG), where

$$\mathcal{X}_i = \mathcal{X}'_i := \{0, 1\}, \qquad i \in \{1, \ldots, 4\}, \qquad U_1 = U_2 := \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\boldsymbol{\psi} := \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad \rho := \boldsymbol{\psi} \cdot \boldsymbol{\psi}^{\mathsf{H}}.$$



# SQMF for Hardy's Paradox

The following matrix shows the components of the SQMF $q(\boldsymbol{x}, \boldsymbol{x}')$, where both the row index $(x_1, \ldots, x_4)$ and column index $(x_1', \ldots, x_4')$ range over $(0,0,0,0), (0,0,0,1), \ldots, (1,1,1,1)$.

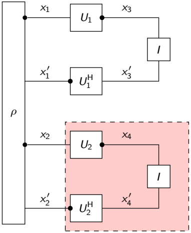
$$
\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

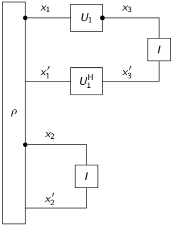
Here: $\alpha_1 := 0.0833$, $\beta_1 := -0.0833$.

Note that the above matrix is not diagonal.

# SQMF for Hardy's Paradox

Consider $q_{1,2,3}(x_1, x_2, x_3, x_1', x_2', x_3') := \sum_{x_4, x_4'} q(x_1, \ldots, x_4, x_1', \ldots, x_4')$.




$$\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

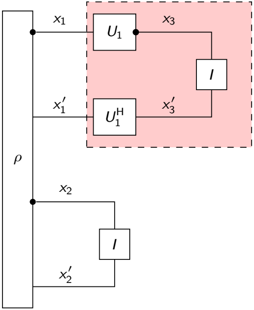
$\alpha_1 := 0.0833,\ \beta_1 := -0.0833$

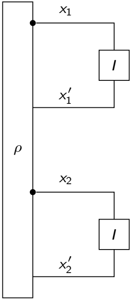
$$\begin{pmatrix}
\alpha_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 & 0 & \beta_2 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\
0 & \beta_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

$\alpha_2 := 0.1667,\ \beta_2 := -0.1667$

# SQMF for Hardy's Paradox

Consider $q_{1,2}(x_1, x_2, x_1', x_2') := \sum_{x_3, x_3'} q_{1,2,3}(x_1, x_2, x_3, x_1', x_2', x_3')$.



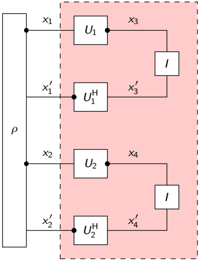

$$\begin{pmatrix} \alpha_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.\ ,$$

$\alpha_2 := 0.1667,\ \beta_2 := -0.1667$

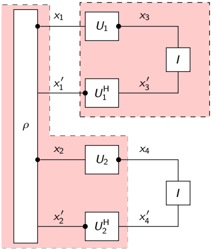
$$\frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that the above matrix is diagonal.

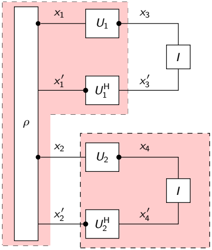
# SQMF for Hardy's Paradox



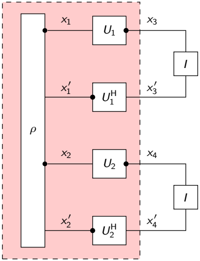

$$q_{1,2} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$




$$q_{1,4} = \frac{1}{6}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

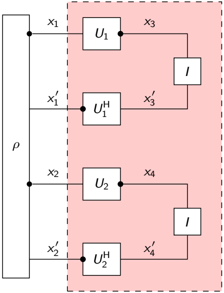



$$q_{3,2} = \frac{1}{6}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

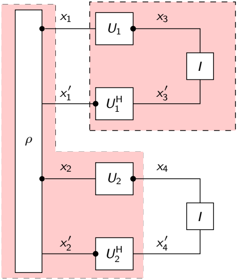



$$q_{3,4} = \frac{1}{12}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

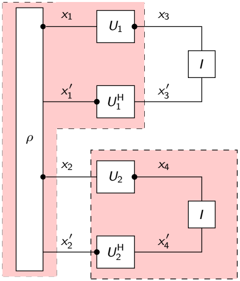
# SQMF for Hardy's Paradox



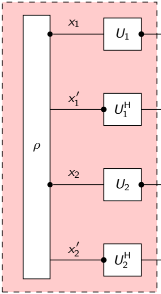

$$p_{1,2} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$




$$p_{1,4} = \frac{1}{6}\begin{pmatrix} 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$




$$p_{3,2} = \frac{1}{6}\begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$




$$p_{3,4} = \frac{1}{12}\begin{pmatrix} 9 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

# SQMF for Hardy's Paradox

- The marginal $p_{3,4}(1,1) = \frac{1}{12}$ shows that it is possible to have $x_3 = x_4 = 1$.
- The marginals $p_{3,2}(1,0) = 0$ and $p_{3,2}(1,1) = 1/6$ show that the condition $x_3 = 1$ implies $x_2 = 1$.
- The marginals $p_{1,4}(0,1) = 0$ and $p_{1,4}(1,1) = 1/6$ show that the condition $x_4 = 1$ implies $x_1 = 1$.
- However, the marginal $p_{1,2}(1,1) = 0$ implies that we cannot have $x_1 = x_2 = 1$, which contradicts $p_{3,4}(1,1) > 0$.

# SQMF for Hardy's Paradox

## Remark

- *We have expressed Hardy's paradox in terms of marginals of SQMFs.*
- *Other paradoxes (e.g. Bell's test, Wigner's friend experiment, and the Frauchiger-Renner paradox) can also be expressed in terms of some suitably defined SQMFs.*

# SQMFs

## Remark

- *For any two sets $\mathcal{I}_1,\ \mathcal{I}_2 \in \mathcal{K}$, the following local consistency constraint holds:*

$$\sum_{\boldsymbol{x}_{\mathcal{I}_1 \setminus \mathcal{I}_2}} p(\boldsymbol{x}_{\mathcal{I}_1}) = \sum_{\boldsymbol{x}_{\mathcal{I}_2 \setminus \mathcal{I}_1}} p(\boldsymbol{x}_{\mathcal{I}_2}) \quad \textit{(for all } \boldsymbol{x}_{\mathcal{I}_1 \cap \mathcal{I}_2}\textit{).}$$

- *Interestingly, this requirement is very similar to the properties of the beliefs in the local marginal polytope (LMP) of a standard factor graph (S-FG) [Wainwright and Jordan, 2008].*

# SQMFs

There are two extreme cases to be considered:

1. The set of marginals $\{p_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{K}}$ can be achieved by some joint PMF.
2. The set of marginals $\{p_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{K}}$ satisfies only the local consistency constraints, i.e., $\{p_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{K}}$ is in the LMP.

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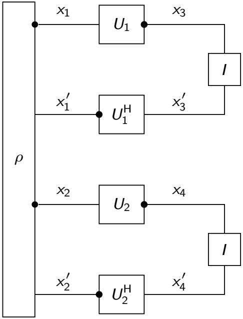
Consider the set of marginals created by jointly classicable variables in the following quantum system.




How general can the marginals $\{p_{\mathcal{I}}\}_{\mathcal{I}\in\mathcal{K}}$ be for different $\rho$, $U_1$, and $U_2$?

# Main Results

# Main Results

## Definition

We define

1. $\mathcal{M}(\mathsf{N})$ to be the set of realizable marginals of the S-FG $\mathsf{N} \in \{\mathsf{N}_1, \mathsf{N}_2, \mathsf{N}_3\}$, where the local functions in N are varied;
2. $\mathcal{LM}(\mathcal{K})$ to be the LMP of the S-FG $\mathsf{N}_1$;
3. $\mathcal{M}(\mathsf{N}_4)$ to be the set of the classicable variables' marginals in the two-qubit system $\mathsf{N}_4$, where $\rho$, $U_1$, and $U_2$ are varied.

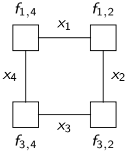



**Figure:** The S-FG $\mathsf{N}_1$.

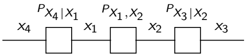



**Figure:** The S-FG $\mathsf{N}_2$.

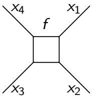



**Figure:** The S-FG $\mathsf{N}_3$.

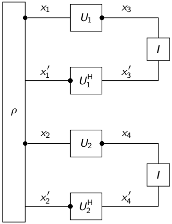
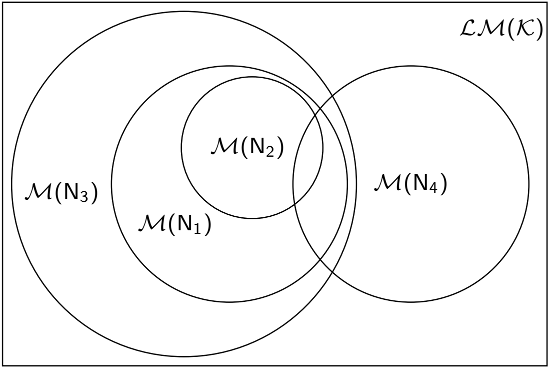



**Figure:** The Q-FG $\mathsf{N}_4$.

# Main Results

## Theorem

*The following Venn diagram holds.*




*We prove that each part in the diagram is non-empty.*

# Main Results

Let us consider the random variables $X_1, \ldots, X_4 \in \{0, 1\}$. The Clauser-Horne-Shimony-Holt (CHSH) inequality states that

$$\left|\mathbb{E}(X_1 \cdot X_2) + \mathbb{E}(X_1 \cdot X_4) + \mathbb{E}(X_3 \cdot X_2) - \mathbb{E}(X_3 \cdot X_4)\right| \leq 2.$$

In this paper, we prove a Pearson correlation coefficient (PCC)-based variant of the CHSH inequality.

### Theorem

*Suppose that the random variables $X_1, \ldots, X_4 \in \{0, 1\}$ satisfy*

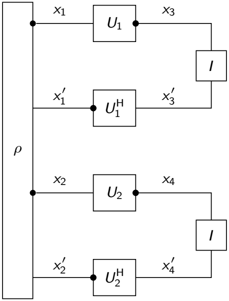
$$\mathrm{Var}(X_1),\ \ldots,\ \mathrm{Var}(X_4) \in \mathbb{R}_{>0}.$$

*Then the following PCC-based CHSH inequality holds:*

$$\left|\mathrm{Corr}(X_1 \cdot X_2) + \mathrm{Corr}(X_1 \cdot X_4) + \mathrm{Corr}(X_3 \cdot X_2) - \mathrm{Corr}(X_3 \cdot X_4)\right| \leq \frac{5}{2}.$$

*This resolves a conjecture proposed in [Pozsgay et al., 2017].*

# Conclusion




## Proposition

- *The SQMF of the above Q-FG can lead to "incompatible" marginals.*
- *We characterize the relationships among the sets of marginals mentioned in the previous slides.*
- *Many well-known quantum phenomena, e.g., Hardy's paradox and Bell's test, can be cast with this SQMF.*

# Selected References I


- Dowker, H. F. and Halliwell, J. J. (1992).
  Quantum mechanics of history: The decoherence functional in quantum mechanics.
  *Phys. Rev. D*, 46:1580–1609.

- Gell-Mann, M. and Hartle, J. B. (1989).
  quantum mechanics in the light of quantum cosmology.
  In *Proc. Santa Fe Institute Workshop on Com-plexity, Entropy, and the Physics of Information*.

- Griffiths, R. B. (2002).
  *Consistent Quantum Theory*.
  Cambridge Univ. Press.

- Loeliger, H. and Vontobel, P. O. (2020).
  Quantum measurement as marginalization and nested quantum systems.
  *IEEE Trans. Inf. Theory*, 66(6):3485–3499.

# Selected References II


Loeliger, H.-A. and Vontobel, P. O. (2017).
Factor graphs for quantum probabilities.
*IEEE Trans. Inf. Theory*, 63(9):5642–5665.

Pozsgay, V., Hirsch, F., Branciard, C., and Brunner, N. (2017).
Covariance Bell inequalities.
*Phys. Rev. A*, 96:062128.

Wainwright, M. J. and Jordan, M. I. (2008).
Graphical models, exponential families, and variational inference.
*Foundation and Trends Machine Learning*, 1(1–2):1–305.

# Thank you!

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