The Bethe Approximation for Binary Contingency Table Counting and Nonnegative Matrix Permanents

Yuwen Huang

Department of Computer Science and Engineering
The Chinese University of Hong Kong
yuwen.huang@ieee.org

Online seminar talk at TU-Dortmund

February 11, 2025

The Bethe approximation for binary contingency table counting and nonnegative matrix permanents

Overview

Topic 1

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Topic 2

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

The Bethe approximation for binary contingency table counting and nonnegative matrix permanents

- **Overview**

Topic 1

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Topic 2

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Overview

Consider a **standard factor graph (S-FG)** N where **each local function** is defined based on a (possibly different) **multi-affine homogeneous real stable** (MAHRS) polynomial.

Various **fundamental combinatorial problems** in the complexity class **#P-complete**, *e.g.*,

1. **counting** the number of **binary contingency tables** with prescribed marginals
2. and computing the **permanent** of a **non-negative square matrix**

can be **reformulated** as the problem of computing the **partition function** of the S-FG.

Overview

Graphical-model-based approximation of the partition function

1. Consider an **arbitrary instance** S-FG N of this class of S-FGs.
   The **partition function** is $Z(\mathsf{N})$.

2. Run the **sum-product algorithm (SPA)**, a.k.a. **belief propagation (BP)**, on N to get the **Bethe approximation of partition function**, *i.e.*, the Bethe partition function

$$Z_{\mathrm{B}}(\mathsf{N}) \triangleq \exp\left(- \min_{\boldsymbol{\beta} \in \mathcal{L}(\mathsf{N})} F_{\mathrm{B}}(\boldsymbol{\beta}) \right),$$

   where (more details later)
   - $\mathcal{L}(\mathsf{N})$ is the **local marginal polytope (LMP)**;
   - F_{B} is the **Bethe free energy function**.

Overview of Topic 1

We focus on **Topic 1** first.

We **prove** that

1. The **projection** of the **local marginal polytope** (LMP) $\mathcal{L}(\mathsf{N})$ on the **edges** in N equals the **convex hull** of the set of **valid configurations** $\mathrm{conv}(\mathcal{C})$.

2. For the **typical** case where the S-FG has a **sum-product algorithm (SPA) fixed point** consisting of **positive-valued messages only**, the SPA finds the value of $Z_{\mathrm{B}}(\mathsf{N})$ **exponentially fast**.

3. The **Bethe free energy function** F_{B} has some **convexity properties**.

Overview of Topic 2

We turn to **Topic 2**.

Consider the matrix

$$\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{n \times n}.$$

Computing $\mathrm{perm}(\boldsymbol{\theta})$, the **matrix permanent** of $\boldsymbol{\theta}$,

is a **#P-complete** problem, **even** in the case where $\boldsymbol{\theta} \in \{0, 1\}^{n \times n}$.

Graphical-model-based approximation:

1. By suitably defining the **multi-affine homogeneous real stable** (MAHRS) **polynomials** in the S-FG, we let the **partition function** $Z(\mathsf{N})$ equals $\mathrm{perm}(\boldsymbol{\theta})$.

2. Run the **sum-product algorithm (SPA)**, a.k.a. **belief propagation (BP)**, on N to get the **Bethe approximation** $\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})$.

Overview of Topic 2

Known bounds (more details later):

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})} \leq 2^{n/2}.$$

Our main results

We **prove** that

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})} < \left(2^{n/2}\right)^{\frac{M-1}{M}}, \qquad M \in \mathbb{Z}_{\geq 1},$$

where $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$ is the **degree-M Bethe permanent**, defined based on **finite graph covers**.

The **lower** bound resolves a **conjecture** in [Vontobel, 2013a].

As $M \to \infty$, we **recover the known bounds**.

The Bethe approximation for binary contingency table counting and nonnegative matrix permanents

Overview

- **Topic 1**

  The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Topic 2

Finite-graph-covers-based bounds for the permanent
of a non-negative square matrix

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

- **An introductory example**

A setup based on binary matrices with prescribed row sums and column sums

Graphical-model-based approximation method

Main results

A more general setup

Main results for a more general setup

Numerical results

Future works and connections to other works

An introductory example

Consider the set of all **binary** 3×3 **matrices**.

We want to know the number of **binary** 3×3 matrices with **row sums** and **column sums** equaling **two**.

The following are **example binary** 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

An introductory example

Consider the set of all **binary** 3×3 **matrices**.

We want to know the number of **binary** 3×3 matrices with **row sums** and **column sums** equaling **two**.

The following are **example binary** 3×3 matrices:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\times}, \quad \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{\checkmark}, \quad \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\checkmark}.$$

The number of such matrices is **3!**.

An introductory example

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- These binary matrices can be viewed as **binary contingency tables** of size 3×3 with **row sums** and **column sums** equaling **two**.
- The number of such **binary contingency tables** is $3!$.

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Setup

Definition

1. $[n] \triangleq \{1, 2, \ldots, n\}$ for $n \in \mathbb{Z}_{\geq 1}$ and $[m] \triangleq \{1, 2, \ldots, m\}$ for $m \in \mathbb{Z}_{\geq 1}$.
2. $\boldsymbol{\gamma} = \big(\gamma(i,j)\big)_{i \in [n], j \in [m]}$: a **$\{0,1\}$-valued matrix** of size $n \times m$.
3. For the **i-th row** $\boldsymbol{\gamma}(i,:)$, we introduce an integer r_i and impose a **constraint** on the **row sum**:

$$\mathcal{X}_{r_i} = \left\{ \gamma(i,:) \;\middle|\; \sum_{j \in [m]} \gamma(i,j) = r_i \right\}.$$

4. For the **j-th column** $\boldsymbol{\gamma}(:,j)$, we introduce an integer c_j and impose a **constraint** on the **column sum**:

$$\mathcal{X}_{c_j} = \left\{ \gamma(:,j) \;\middle|\; \sum_{i \in [n]} \gamma(i,j) = c_j \right\}.$$

Setup

Definition

5. The set of **valid configurations** is defined to be

$$\mathcal{C} \triangleq \left\{ \gamma \in \{0,1\}^{n\times n} \;\middle|\; \begin{array}{c} \gamma(i,:) \in \mathcal{X}_{r_i},\ \forall i \in [n], \\ \gamma(:,j) \in \mathcal{X}_{c_j},\ \forall j \in [m] \end{array} \right\},$$

the set of **binary matrices** such that the ***i*-th row sum** is r_i and the ***j*-th column sum** is c_j.

6. We want to compute the number of the **valid configurations** $|\mathcal{C}|$.

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Graphical-model-based approximation method

Main idea

1. Define a **standard factor graph (S-FG)** N whose partition function equals

$$Z(\mathsf{N}) = |\mathcal{C}|.$$

2. Run the **sum product algorithm (SPA)**, a.k.a. **belief propagation** (BP), on the S-FG N to compute the **Bethe approximation of** $|\mathcal{C}|$, denoted by $Z_{\mathrm{B}}(\mathsf{N})$.

Graphical-model-based approximation method

Example

Consider $n = m = 3$ **and** $r_i = c_j = 2$, *i.e.*, $\boldsymbol{\gamma} \in \{0,1\}^{3\times 3}$.

The i**-th row** $\boldsymbol{\gamma}(i,:) \in \mathcal{X}_{r_i}$ and the j**-th column** $\boldsymbol{\gamma}(:,j) \in \mathcal{X}_{c_j}$, where

$$\mathcal{X}_{r_i} = \{(1,1,0),(0,1,1),(1,0,1)\}, \quad \mathcal{X}_{c_j} = \{(1,1,0)^{\mathsf{T}},(0,1,1)^{\mathsf{T}},(1,0,1)^{\mathsf{T}}\}.$$

1. The **local functions**:

$$f_{\mathrm{l},i}\big(\boldsymbol{\gamma}(i,:)\big) \triangleq \begin{cases} 1 & \text{if } \boldsymbol{\gamma}(i,:) \in \mathcal{X}_{r_i} \\ 0 & \text{otherwise} \end{cases}, \quad f_{\mathrm{r},j}\big(\boldsymbol{\gamma}(:,j)\big) \triangleq \begin{cases} 1 & \text{if } \boldsymbol{\gamma}(:,j) \in \mathcal{X}_{c_j} \\ 0 & \text{otherwise} \end{cases}.$$

2. The **support** of the local functions:

$$\mathcal{X}_{f_{\mathrm{l},i}} \triangleq \big\{\boldsymbol{\gamma}(i,:) \in \{0,1\}^3 \mid f_{\mathrm{l},i}\big(\boldsymbol{\gamma}(i,:)\big) > 0\big\} = \mathcal{X}_{r_i},$$
$$\mathcal{X}_{f_{\mathrm{r},j}} \triangleq \big\{\boldsymbol{\gamma}(:,j) \in \{0,1\}^3 \mid f_{\mathrm{r},j}\big(\boldsymbol{\gamma}(:,j)\big) > 0\big\} = \mathcal{X}_{c_j}.$$

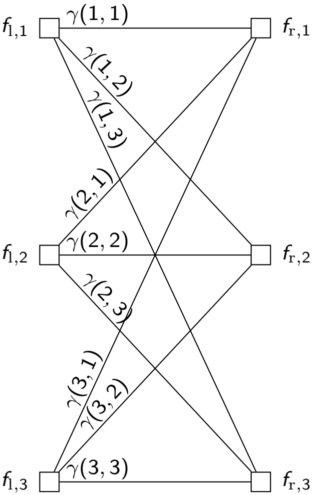
Graphical-model-based approximation method

3. The $\{0,1\}$-valued **global function**:

$$g(\boldsymbol{\gamma}) \triangleq f_{\mathrm{l},1}\big(\gamma(1,1),\gamma(1,2),\gamma(1,3)\big) \cdot f_{\mathrm{l},2}\big(\gamma(2,1),\gamma(2,2),\gamma(2,3)\big) \cdots f_{\mathrm{r},2}\big(\gamma(1,2),\gamma(2,2),\gamma(3,2)\big) \cdot f_{\mathrm{r},3}\big(\gamma(1,3),\gamma(2,3),\gamma(3,3)\big).$$

The **previously defined** set of **valid configurations** is equal to the **support** of the global function:

$$\mathcal{C} = \left\{\boldsymbol{\gamma} \in \{0,1\}^{3\times 3} \;\middle|\; g(\boldsymbol{\gamma}) > 0\right\}.$$



4. The **partition function**:

$$Z(\mathsf{N}) \triangleq \sum_{\boldsymbol{\gamma}\in\{0,1\}^{3\times 3}} g(\boldsymbol{\gamma}) = |\mathcal{C}|.$$

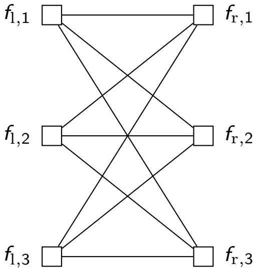
Graphical-model-based approximation method

5. The **Bethe approximation** of the partition function, *i.e.*, the **Bethe partition function**, is defined to be

$$Z_{\mathrm{B}}(\mathsf{N}) \triangleq \exp\left(- \min_{\boldsymbol{\beta} \in \mathcal{L}(\mathsf{N})} F_{\mathrm{B}}(\boldsymbol{\beta}) \right),$$

where F_{B} is the **Bethe free energy (BFE)** function,

where $\mathcal{L}(\mathsf{N})$ is the **local marginal polytope** (LMP) (see, *e.g.*, [Wainwright and Jordan, 2008]).



6. Then we run the **sum-product algorithm (SPA)**, a.k.a. **belief propagation (BP)**, on the S-FG N to get $Z_{\mathrm{B}}(\mathsf{N})$.

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Our main contribution for Topic 1

1. The **projection** of the **LMP** on the **edges** in N **equals** conv($\mathcal{C}$).
   (For general S-FGs, this projection is a **relaxation** of conv($\mathcal{C}$), *i.e.*, conv($\mathcal{C}$) is a **strict subset** of this **projection**.)

2. For the **typical case** where N has an **SPA fixed point** consisting of **positive-valued messages only**, the SPA finds the value of Z_{B}(N) **exponentially fast**.

3. The **BFE function** has some **convexity properties**.

Comments

- A **generalization** of parts of the results in [Vontobel, 2013a].
- Even though the S-FG has a **non-trivial cyclic structure**, the SPA has **a good performance**.

Our main contribution for Topic 1

Comments

For the setup where $n = m$, $r_i = 1$, and $c_j = 1$, it holds that

- $\mathcal{C} = \{\gamma \mid \gamma \text{ is a } \textbf{permutation matrix} \text{ of size } n\text{-by-}n\}$
- The **projection** of the **LMP** on the **edges** equals the set of **doubly stochastic matrices** of size n-by-n.

Birkhoff–von Neumann theorem

The set of **doubly stochastic matrices** of size n-by-n is the **convex hull** of the set of the **permutation matrices** of size n-by-n.

The main result that $\mathrm{conv}(\mathcal{C})$ equals the **projection** of the **LMP** on the **edges** for our considered S-FG, can be viewed as a **generalization**.

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

A more general setup

An example S-FG

Consider $n = m = 3$ and $r_i = c_j = 2$. Then

$$f_{1,i}\big(\gamma(i,:)\big) = \begin{cases} 1 & \text{if } \gamma(i,:) \in \{(1,1,0),(0,1,1),(1,0,1)\} \\ 0 & \text{otherwise} \end{cases},$$

which corresponds to a **multi-affine homogeneous real stable (MAHRS) polynomial** w.r.t. the **indeterminates** in $\boldsymbol{L} \triangleq \big(L_1, L_2, L_3\big) \in \mathbb{C}^3$:

$$\begin{aligned} p_i(\boldsymbol{L}) &= \sum_{\gamma(i,:)\in\{0,1\}^3} f_{1,i}\big(\gamma(i,:)\big) \cdot \prod_{j\in[3]} \big(L_j\big)^{\gamma(i,j)} \\ &= L_1 \cdot L_2 + L_2 \cdot L_3 + L_1 \cdot L_3, \end{aligned}$$

Remark

- For details of **real stable polynomials**, see, *e.g.*, [Gharan, 2020]

1. **Start** from the problem of **counting contigency tables**.
2. Define the S-FG based on this **counting problem**.
3. Observe that each **local functions** corresponds to a **special** MAHRS polynomial.

Consider a **more general** setup where **each local function** is defined based on a (possibly different) arbitrary MAHRS polynomial.

Do the previous results **hold** in this **more general setup**?

Yes!

An MAHRS Polynomials-based S-FG

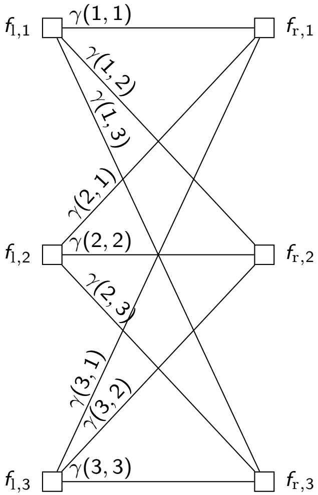
The standard factor graph (S-FG) N consists of

1. **edges**: $(1,1),(1,2),\ldots,(3,3)$;
2. **Binary** matrix

$$\boldsymbol{\gamma} \triangleq \begin{pmatrix} \gamma(1,1) & \gamma(1,2) & \gamma(1,3) \\ \gamma(2,1) & \gamma(2,2) & \gamma(2,3) \\ \gamma(3,1) & \gamma(3,2) & \gamma(3,3) \end{pmatrix}.$$

3. **Nonnegative-valued** local functions

   $f_{\mathrm{l},1},\ldots,f_{\mathrm{r},3}$;



An MAHRS Polynomials-based S-FG

6. The **local function** $f_{\mathrm{l},i}$ on the **LHS** is defined to be the mapping:

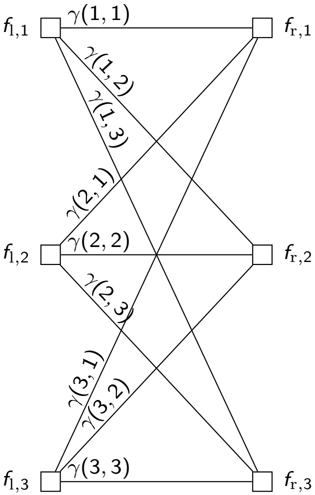
$$\{0,1\}^3 \to \mathbb{R}_{\geq 0}, \quad \boldsymbol{\gamma}(i,:) \mapsto f_{\mathrm{l},i}\big(\boldsymbol{\gamma}(i,:)\big)$$

such that it corresponds to an **MAHRS polynomial**.

7. The **support** of $f_{\mathrm{l},i}$:

$$\mathcal{X}_{f_{\mathrm{l},i}} \triangleq \left\{ \boldsymbol{\gamma}(i,:) \in \{0,1\}^3 \;\middle|\; f_{\mathrm{l},i}\big(\boldsymbol{\gamma}(i,:)\big) > 0 \right\}.$$

8. A **similar idea** in the definitions of $f_{\mathrm{r},j}$ and $\mathcal{X}_{f_{\mathrm{r},j}}$ on the **RHS**.



An MAHRS Polynomials-based S-FG

9. The **nonnegative-valued global function**:

$$g(\gamma) \triangleq f_{\mathrm{l},1}\big(\gamma(1,:)\big) \cdot f_{\mathrm{l},2}\big(\gamma(2,:)\big) \cdot f_{\mathrm{l},3}\big(\gamma(3,:)\big) \cdot f_{\mathrm{r},1}\big(\gamma(:,1)\big) \cdot f_{\mathrm{r},2}\big(\gamma(:,2)\big) \cdot f_{\mathrm{r},3}\big(\gamma(:,3)\big).$$

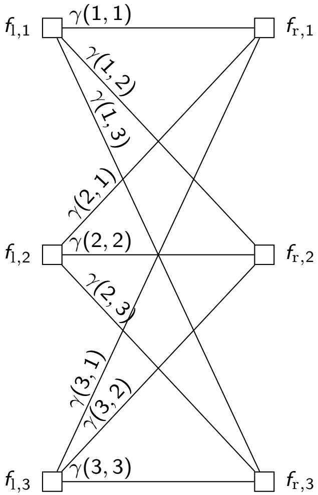
10. The set of **valid configurations**:

$$\mathcal{C} \triangleq \left\{\gamma \in \{0,1\}^{3\times 3} \;\middle|\; g(\gamma) > 0\right\},$$

which is also the **support** of the **global function**.

11. The **partition function**:

$$Z(\mathsf{N}) \triangleq \sum_{\gamma\in\mathcal{C}} g(\gamma).$$



The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Known results

Consider an **S-FG** N where **each local function** is defined based on a (possibly different) **MAHRS** polynomial.

Remarks

- Exactly computing $Z(\mathsf{N})$ is a **#P-complete problem** in general.
- **Run the SPA** to find the value of the **Bethe partition function** $Z_{\mathrm{B}}(\mathsf{N})$ that **approximates** $Z(\mathsf{N})$.
- [Straszak and Vishnoi, 2019, Theorem 3.2]: $Z_{\mathrm{B}}(\mathsf{N}) \leq Z(\mathsf{N})$.
- **Other real-stable-polynomial-based approximation of** $Z(\mathsf{N})$ [Gurvits, 2015, Brändén et al., 2023].

Our main contribution for Topic 1

Consider an **S-FG** N where **each local function** is defined based on a (possibly different) **MAHRS** polynomial.

- The **support** $\mathcal{X}_{f_{\mathrm{l},i}}$ on the LHS corresponds to **a set of bases of a matroid** [Brändén, 2007].
- The support of the **product** of the **local functions** on the **LHS** is $\{\mathcal{X}_{f_{\mathrm{l},1}} \times \mathcal{X}_{f_{\mathrm{l},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{l},n}}\}$.
- Similarly for the local functions and the **support** on the **RHS**.
- The support of the **global function** equals the **intersection** of the bases of **matroids**:

$$\mathcal{C} = \{\mathcal{X}_{f_{\mathrm{l},1}} \times \mathcal{X}_{f_{\mathrm{l},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{l},n}}\} \bigcap \{\mathcal{X}_{f_{\mathrm{r},1}} \times \mathcal{X}_{f_{\mathrm{r},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{r},m}}\}$$

Our main contribution for Topic 1

1. The **convex hull** $\mathrm{conv}(\mathcal{C})$ is the **projection of the LMP** on the **edges**.
   (Based on results on intersection of matroids [Oxley, 2011].)

2. For the typical case where the S-FG has an SPA **fixed point** consisting of **positive-valued messages only**, the SPA finds the value of $Z_{\mathrm{B}}(\mathsf{N})$ **exponentially fast**.
   (Based on the properties of **real stable polynomials** in [Brändén, 2007].)

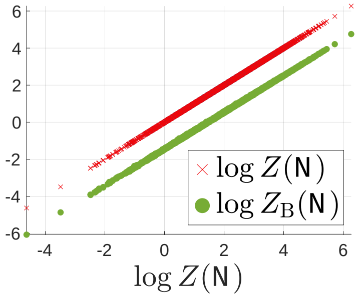
3. The **Bethe free energy function** F_{B} has some **convexity properties**.
   The proof of the convexity is **new**.
   (Based on the **dual** form of $Z_{\mathrm{B}}(\mathsf{N})$ in [Straszak and Vishnoi, 2019, Anari and Gharan, 2021].)

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Numerical results

Setup

- We first consider the case $n = m = 6$ **and** $r_i = c_j = 2$, *i.e.*, each local function is defined based on a (possibly different) **MAHRS polynomial** having 6 **indeterminates and degree** 2.



- We independently randomly generate 3000 **instances** of N.

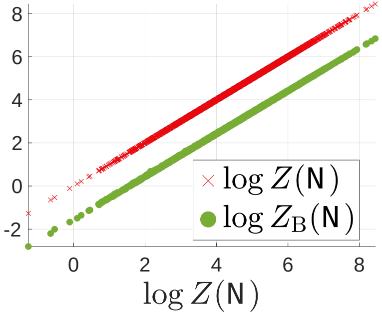
Observation

- $Z_{\mathrm{B}}(\mathrm{N}) \leq Z(\mathrm{N})$ ([Straszak and Vishnoi, 2019, Theroem 3.2]).
- $Z_{\mathrm{B}}(\mathrm{N})$ provides a **good estimate** of $Z(\mathrm{N})$ in this case.

Numerical results

Setup

Consider **the same setup** as the previous case, but with $n = m = 6$ **replaced** by $n = m = 7$.



Observation

We can make **similar observations**.

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

Future work

- Consider a **more general** S-FG, where each local function corresponds to a **more general** polynomial.
- Prove the **convergence** of the SPA for a **more general** S-FG.

Connection to other works

- **Polynomial approaches** to approximate **partition functions**.
  [Gurvits, 2011, Straszak and Vishnoi, 2017, Anari and Gharan, 2021]

- The properties of **real stable** polynomials and the **partition functions**.
  [Brändén, 2014, Borcea and Brändén, 2009, Borcea et al., 2009]

The Bethe approximation for binary contingency table counting and nonnegative matrix permanents

Overview

Topic 1

The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials

- **Topic 2**

  Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Setup

- $[n] \triangleq \{1, 2, \ldots, n\}$.
- $\boldsymbol{\theta} \triangleq \big(\theta(i,j)\big)_{i,j\in[n]} \in \mathbb{R}_{\geq 0}^{n\times n}$: **a non-negative real-valued matrix**.
- $\mathcal{S}_{[n]}$ is the set of all $n!$ **permutations** of $[n]$.
- The **determinant**:

$$\det(\boldsymbol{\theta}) \triangleq \sum_{\sigma\in\mathcal{S}_{[n]}} \mathrm{sgn}(\sigma) \cdot \prod_{i\in[n]} \theta\big(i,\sigma(i)\big).$$

  The **complexity** of **evaluating** $\det(\boldsymbol{\theta})$ is $O(n^3)$.

- The **permanent**:

$$\mathrm{perm}(\boldsymbol{\theta}) \triangleq \sum_{\sigma\in\mathcal{S}_{[n]}} \prod_{i\in[n]} \theta\big(i,\sigma(i)\big).$$

  The **complexity** class of **evaluating** $\mathrm{perm}(\boldsymbol{\theta})$ is **#P**-complete.

Note: In the following, we consider **nonnegative-valued square** matrices.

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Graphical-model-based approximation method

1. By suitably defining the **multi-affine homogeneous real stable (MAHRS) polynomials** in the S-FG, we let the **partition function** $Z(\mathsf{N})$ equals $\mathrm{perm}(\boldsymbol{\theta})$.

2. **Reformulate** $Z(\mathsf{N})$:

$$Z(\mathsf{N}) = \mathrm{perm}(\boldsymbol{\theta}) = \exp\left(- \min_{\boldsymbol{p} \in \Pi_{\mathcal{A}(\boldsymbol{\theta})}} F_{\mathrm{G},\boldsymbol{\theta}}(\boldsymbol{p}) \right),$$

   where $F_{\mathrm{G},\boldsymbol{\theta}}$ is the **Gibbs free energy function**.

3. Develop the **Bethe approximation**:

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \triangleq \exp\left(- \min_{\gamma \in \Gamma_n} F_{\mathrm{B},\boldsymbol{\theta}}(\gamma) \right),$$

   where $F_{\mathrm{B},\boldsymbol{\theta}}$ is the **Bethe free energy function**.

An S-FG representation of the permanent

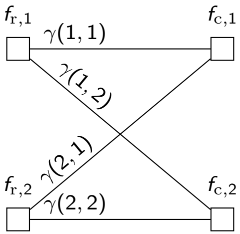
The **standard factor graph (S-FG)** N for $\boldsymbol{\theta}$ consists of

$$\boldsymbol{\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. **edges**: $(1,1),(1,2),(2,1),(2,2)$;
2. **variables** in the matrix

$$\boldsymbol{\gamma} \triangleq \begin{pmatrix} \gamma(1,1) & \gamma(1,2) \\ \gamma(2,1) & \gamma(2,2) \end{pmatrix} \in \{0,1\}^{2\times 2}.$$

3. **nonnegative-valued** local functions
   $f_{\mathrm{r},1}$, $f_{\mathrm{r},2}$, and $f_{\mathrm{c},1}$, $f_{\mathrm{c},2}$;



An S-FG representation of the permanent

The details of the **standard factor graph** (S-FG) N for $\boldsymbol{\theta}$ are as follows:

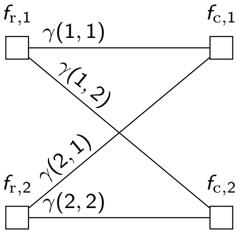
1. The **global function**:

$$g(\gamma) \triangleq f_{\mathrm{r},1}\big(\gamma(1,:)\big) \cdot f_{\mathrm{r},2}\big(\gamma(2,:)\big) \cdot f_{\mathrm{c},1}\big(\gamma(:,1)\big) \cdot f_{\mathrm{c},2}\big(\gamma(:,2)\big);$$

$$\boldsymbol{\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2. The **partition function**:

$$Z(\mathsf{N}) = \sum_{\gamma \in \{0,1\}^{2\times 2}} g(\gamma) = a \cdot d + b \cdot c = \mathrm{perm}(\boldsymbol{\theta}).$$

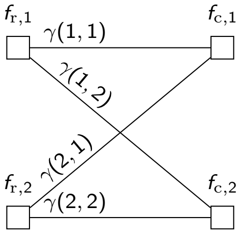


Graphical-model-based approximation method

3. [Vontobel, 2013a]

   The **Bethe approximation** of the permanent, *i.e.*, the **Bethe partition function**:

$$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \triangleq \exp\left(- \min_{\boldsymbol{\gamma} \in \Gamma_n} F_{\mathrm{B},\boldsymbol{\theta}}(\gamma) \right),$$



where $F_{\mathrm{B},\boldsymbol{\theta}}$ is the **Bethe free energy (BFE)** function,

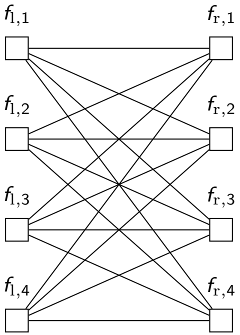
where Γ_n is the set of **doubly stochastic matrices** of size $n \times n$.

Note that $\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})$ is also called the **Bethe permanent**.

Graphical-model-based approximation method

We can make **similar definitions** for a more general case:

$$\boldsymbol{\theta} = \begin{pmatrix} \theta(1,1) & \cdots & \theta(1,4) \\ \vdots & \ddots & \vdots \\ \theta(4,1) & \cdots & \theta(4,4) \end{pmatrix} \in \mathbb{R}_{\geq 0}^{4\times 4}.$$



The **S-FG** for $\boldsymbol{\theta}$.

Graphical-model-based approximation method

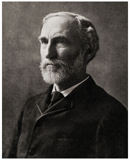
Bounding the **permanent** in terms of the **Bethe permanent**:

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})} \leq 2^{n/2}.$$

- The first inequality was **proven** by [Gurvits, 2011] with the help of an **inequality** by [Schrijver, 1998].
- The second inequality was **conjectured** by [Gurvits, 2011] and **proven** by [Anari and Rezaei, 2019].

[Vontobel, 2013a]
The **sum-product algorithm (SPA)** finds the value of $\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})$ **exponentially fast**.

Graphical-model-based approximation method

	 Josiah W. Gibbs	 Hans Bethe
	Permanent	Bethe permanent
Combinatorial	$\mathrm{perm}(\boldsymbol{\theta}) = \sum\limits_{\sigma \in \mathcal{S}_{[n]}} \prod\limits_{i \in [n]} \theta\big(i, \sigma(i)\big)$ (the sum of weighted configurations)	???
Analytical	$\mathrm{perm}(\boldsymbol{\theta}) = \exp\Big(-\min\limits_{\boldsymbol{p} \in \Pi_{\mathcal{A}(\boldsymbol{\theta})}} F_{\mathrm{G},\boldsymbol{\theta}}(\boldsymbol{p})\Big)$	$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) = \exp\Big(-\min\limits_{\gamma \in \Gamma_n} F_{\mathrm{B},\boldsymbol{\theta}}(\gamma)\Big)$

Use **finite graph covers** to give a **combinatorial characterization**.

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Finite graph covers

Graph covers (a.k.a. graph lifts) have appeared in **various contexts**:

- [Angluin, 1980]:

  Local and global properties in **networks of processors**.

- N. Linial *et al.* (*e.g.*, [Amit and Linial, 2002])

  Various papers on characterizing properties of **graph covers**.

- [Marcus et al., 2015]:

  The **existence** of infinite families of **regular bipartite Ramanujan graphs** of every degree **larger** than 2.

Graph covers in **coding theory**:

- [Koetter and Vontobel, 2003]:

  Analysis of **message-passing iterative decoders** via graph covers.

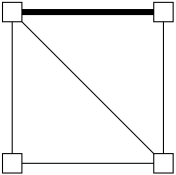
Finite graph covers

Outline

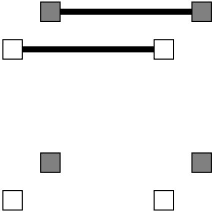
Introduce a **combinatorial characterization** of the **Bethe partition function** proven in [Vontobel, 2013b].

1. Consider **general S-FGs**, **extending** the definition of the S-FG for the **matrix permanent**.
2. Introduce **finite graph covers**.
3. Present a **combinatorial characterization** of the **Bethe partition function** in terms of **finite graph covers**.
4. Discuss a **combinatorial characterization** in the case of the S-FG for the matrix **permanent**.

Finite graph covers



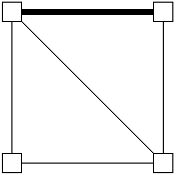
original graph



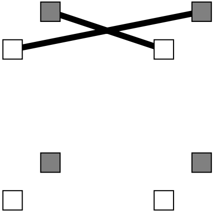
(a possible)

2-cover of original graph

Finite graph covers



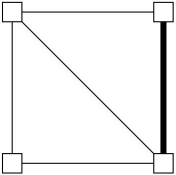
original graph



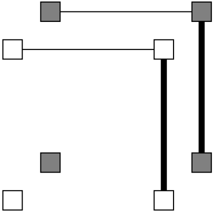
(a possible)

2-cover of original graph

Finite graph covers



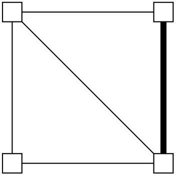
original graph



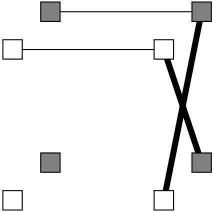
(a possible)

2-cover of original graph

Finite graph covers



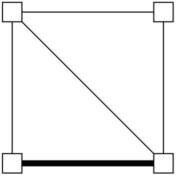
original graph



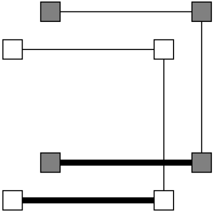
(a possible)

2-cover of original graph

Finite graph covers



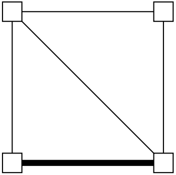
original graph



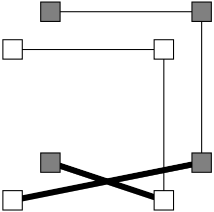
(a possible)

2-cover of original graph

Finite graph covers



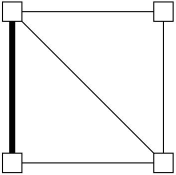
original graph



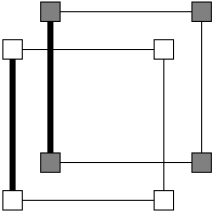
(a possible)

2-cover of original graph

Finite graph covers



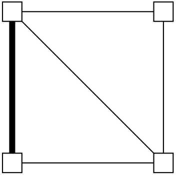
original graph



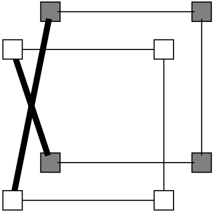
(a possible)

2-cover of original graph

Finite graph covers



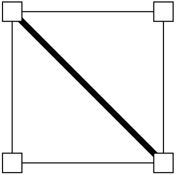
original graph



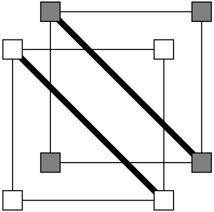
(a possible)

2-cover of original graph

Finite graph covers

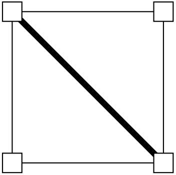


original graph

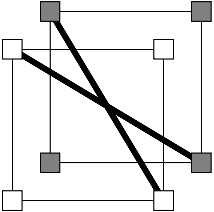


(a possible)
2-cover of original graph

Finite graph covers



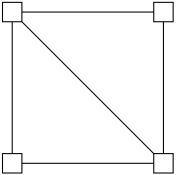
original graph



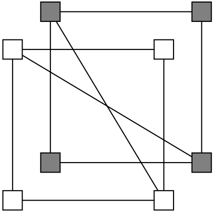
(a possible)

2-cover of original graph

Finite graph covers



original graph



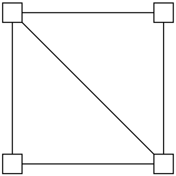
(a possible)

2-cover of original graph

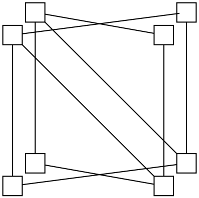
Definition: A graph C is a **double cover** of another graph G if....

Note: the original graph has $2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! = (2!)^5$ **double covers**.

Finite graph covers

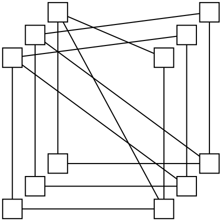


original graph



(a possible)

double cover

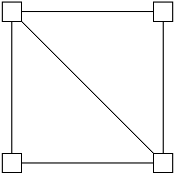


(a possible)

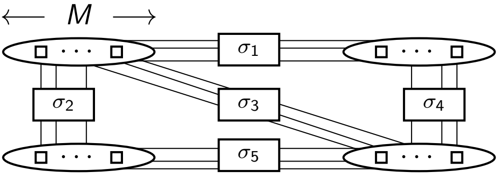
triple cover

Besides double covers, a graph also has many **triple** covers, **quadruple** covers, **quintuple** covers, *etc*.

Finite graph covers



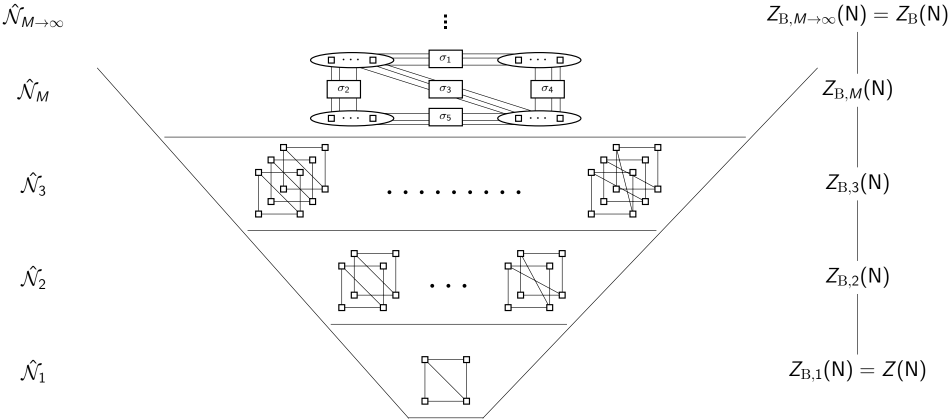
original graph



(a possible) M-fold cover of original graph

An ***M*-fold cover** is also called **a cover of degree *M***.

Do not confuse this degree with the degree of a vertex!



The **degree-M Bethe partition function**:

$$Z_{\mathrm{B},M}(\mathsf{N}) \triangleq \sqrt[M]{\frac{1}{|\hat{\mathcal{N}}_M|} \sum_{\hat{\mathsf{N}} \in \hat{\mathcal{N}}_M} Z(\hat{\mathsf{N}})}.$$

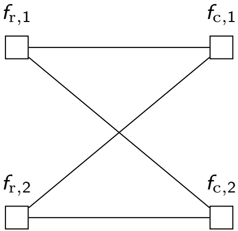
The graph-cover theorem [Vontobel, 2013b]

For **any S-FG** N, it holds that $\limsup_{M\to\infty} Z_{\mathrm{B},M}(\mathsf{N}) = Z_{\mathrm{B}}(\mathsf{N})$.

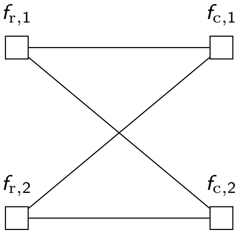
Finite graph covers

Focus on the S-FGs associated with the **matrix permanents**.

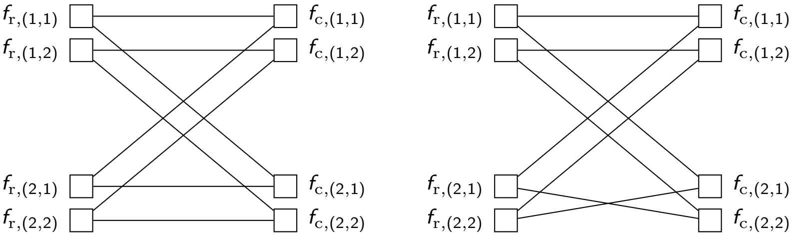
Example



$$\boldsymbol{\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{\geq 0}^{2\times 2}, \qquad Z(\mathsf{N}) = \mathrm{perm}(\boldsymbol{\theta}) = a \cdot d + b \cdot c.$$



Original graph

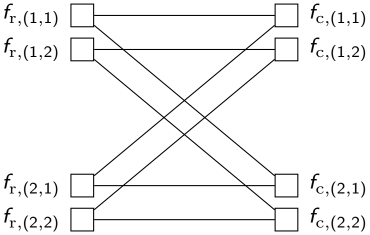


Possible 2-covers

Each 2-cover $\hat{\mathsf{N}}$ is **an S-FG** and induces the **partition function** $Z(\hat{\mathsf{N}})$.

For the 2-cover N on the RHS, **reformulate** $Z(\hat{\mathsf{N}})$:

$$Z(\hat{\mathsf{N}}) = \mathrm{perm}(\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M})$$



where the **P_M-lifting of θ**:

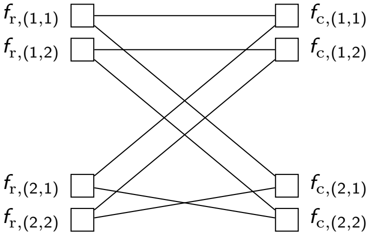
$$\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M} = \left(\begin{array}{c|c} a \cdot \boldsymbol{P}^{(1,1)} & b \cdot \boldsymbol{P}^{(1,2)} \\ \hline c \cdot \boldsymbol{P}^{(2,1)} & d \cdot \boldsymbol{P}^{(2,2)} \end{array}\right) = \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & d & 0 \\ 0 & c & 0 & d \end{array}\right),$$

and $\boldsymbol{P}^{(1,1)}, \ldots, \boldsymbol{P}^{(2,2)}$ are **permutation** matrices:

$$\boldsymbol{P}^{(1,1)} = \boldsymbol{P}^{(1,2)} = \boldsymbol{P}^{(2,1)} = \boldsymbol{P}^{(2,2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the 2-cover N on the RHS, **reformulate** $Z(\hat{\mathsf{N}})$:

$$Z(\hat{\mathsf{N}}) = \mathrm{perm}(\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M})$$



where the **P_M-lifting** of $\boldsymbol{\theta}$:

$$\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M} = \left(\begin{array}{c|c} a \cdot \boldsymbol{P}^{(1,1)} & b \cdot \boldsymbol{P}^{(1,2)} \\ \hline c \cdot \boldsymbol{P}^{(2,1)} & d \cdot \boldsymbol{P}^{(2,2)} \end{array}\right) = \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & a & 0 & b \\ \hline c & 0 & 0 & d \\ 0 & c & d & 0 \end{array}\right),$$

and $\boldsymbol{P}^{(1,1)}, \ldots, \boldsymbol{P}^{(2,2)}$ are **permutation** matrices:

$$\boldsymbol{P}^{(1,1)} = \boldsymbol{P}^{(1,2)} = \boldsymbol{P}^{(2,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{P}^{(2,2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finite graph covers

Analyzing the **degree-M finite graph covers** $\hat{\mathsf{N}}$ is **equivalent** to analyzing the $\boldsymbol{P}_M$**-liftings** of $\boldsymbol{\theta}$.

For general $\boldsymbol{\theta} \in \mathbb{R}^{n \times n}$, define a $\boldsymbol{P}_M$**-lifting** of $\boldsymbol{\theta}$:

$$\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M} \triangleq \begin{pmatrix} \theta(1,1) \cdot \boldsymbol{P}^{(1,1)} & \cdots & \theta(1,n) \cdot \boldsymbol{P}^{(1,n)} \\ \vdots & \ddots & \vdots \\ \theta(n,1) \cdot \boldsymbol{P}^{(n,1)} & \cdots & \theta(n,n) \cdot \boldsymbol{P}^{(n,n)} \end{pmatrix} \in \mathbb{R}_{\geq 0}^{Mn \times Mn},$$

where

- $\boldsymbol{P}_M \triangleq (\boldsymbol{P}^{(i,j)})_{i,j \in [n]}$;
- $\boldsymbol{P}^{(i,j)}$ is a **permutation** matrix of size $M \times M$.

Finite graph covers

Remark

Consider **degree-M of finite graph covers** of N.

1. For each **degree-M finite graph cover** $\hat{\mathsf{N}}$, it is an **S-FG** and induces a **partition function** $Z(\hat{\mathsf{N}})$.

2. Observe that $Z(\hat{\mathsf{N}}) = \mathrm{perm}(\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M})$ for some **$\boldsymbol{P}_M$-lifting** of $\boldsymbol{\theta}$.

Finite graph covers

Definitions

1. The degree-M **Bethe partition function** of N to be

$$Z_{\mathrm{B},M}(\mathsf{N}) \triangleq \sqrt[M]{\frac{1}{|\hat{\mathcal{N}}_M|} \cdot \sum_{\hat{\mathsf{N}} \in \hat{\mathcal{N}}_M} Z(\hat{\mathsf{N}})}.$$

   where $\hat{\mathcal{N}}_M$ is the set of all **M-covers** of N.

2. [Vontobel, 2013a]

   A **reformulation** in terms of the **degree-M Bethe permanent**:

$$\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}) \triangleq \sqrt[M]{\frac{1}{|\tilde{\Psi}_M|} \cdot \sum_{\boldsymbol{P}_M \in \tilde{\Psi}_M} \mathrm{perm}(\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M})}$$

$$= Z_{\mathrm{B},M}(\mathsf{N}),$$

   where the set $\tilde{\Psi}_M$ is **the set of all possible P_M-lifting** of $\boldsymbol{\theta}$.

Finite graph covers

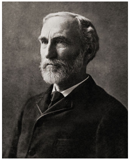
The graph-cover theorem

[Vontobel, 2013a, Vontobel, 2013b]

$$
\begin{array}{ccc}
Z_{\mathrm{B},M}(\mathsf{N})\big|_{M\to\infty} = Z_{\mathrm{B}}(\mathsf{N}) & \qquad & \mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big|_{M\to\infty} = \mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) \\
\Big| & & \Big| \\
Z_{\mathrm{B},M}(\mathsf{N}) & & \mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}) \\
\Big| & & \Big| \\
Z_{\mathrm{B},M}(\mathsf{N})\big|_{M=1} = Z(\mathsf{N}) & & \mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big|_{M=1} = \mathrm{perm}(\boldsymbol{\theta})
\end{array}
$$

A **combinatorial characterization** of the Bethe permanent.

Graphical-model-based approximation method

	 Josiah W. Gibbs	 Hans Bethe
	Permanent	Bethe permanent
Combinatorial	$\mathrm{perm}(\boldsymbol{\theta}) = \sum_{\sigma \in \mathcal{S}_{[n]}} \prod_{i \in [n]} \theta\big(i, \sigma(i)\big)$	$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) = \limsup_{M \to \infty} \mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta}).$
Analytical	$\mathrm{perm}(\boldsymbol{\theta}) = \exp\left(- \min_{\boldsymbol{p} \in \Pi_{\mathcal{A}(\boldsymbol{\theta})}} F_{\mathrm{G},\boldsymbol{\theta}}(\boldsymbol{p}) \right)$	$\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta}) = \exp\left(- \min_{\gamma \in \Gamma_n} F_{\mathrm{B},\boldsymbol{\theta}}(\gamma) \right)$

Our main contribution for Topic 2

We bound $\mathrm{perm}(\boldsymbol{\theta})$ via $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$.

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example ($n = 2$ and $M = 2$)

$$\boldsymbol{\theta} \triangleq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{\geq 0}^{n \times n}.$$

---

1. Γ_n : the set of all **doubly stochastic matrices of size $n \times n$**.
2. $\Gamma_{M,n}$: **the subset of Γ_n** that contains all matrices where the entries are **multiples of $1/M$**.
3. $\boldsymbol{\theta}^{M \cdot \gamma} \triangleq \prod_{i,j \in [n]} \big(\theta(i,j)\big)^{M \cdot \gamma(i,j)}$, for $\gamma \in \Gamma_{M,n}$.

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example continued ($n = 2$ and $M = 2$)

Define

$$\gamma^{(1,0)} \triangleq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{(1,1)} \triangleq \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \gamma^{(0,1)} \triangleq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathrm{perm}(\boldsymbol{\theta}) = a \cdot d + b \cdot c,$$

$$\begin{aligned}\big(\mathrm{perm}(\boldsymbol{\theta})\big)^2 &= \mathbf{1} \cdot (a \cdot d)^2 + \mathbf{2} \cdot a \cdot b \cdot c \cdot d + \mathbf{1} \cdot (c \cdot b)^2 \\ &= \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(1,0)}} + \mathbf{2} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(1,1)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(0,1)}},\end{aligned}$$

$$\begin{aligned}\big(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big)^2 &= \big\langle \mathrm{perm}(\boldsymbol{\theta}^{\uparrow \boldsymbol{P}_M}) \big\rangle_{\boldsymbol{P}_M \in \tilde{\Psi}_M} \\ &= \mathbf{1} \cdot (a \cdot d)^2 + \mathbf{1} \cdot a \cdot b \cdot c \cdot d + \mathbf{1} \cdot (c \cdot b)^2 \\ &= \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(1,0)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(1,1)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \gamma^{(0,1)}}.\end{aligned}$$

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example continued ($n = 2$ and $M = 2$)

$$2 \cdot \left(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\right)^2 = \mathbf{2} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,0)}} + \mathbf{2} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,1)}} + \mathbf{2} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(0,1)}},$$

$$\left(\mathrm{perm}(\boldsymbol{\theta})\right)^2 = \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,0)}} + \mathbf{2} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,1)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(0,1)}},$$

$$\left(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\right)^2 = \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,0)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(1,1)}} + \mathbf{1} \cdot \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}^{(0,1)}}.$$

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example continued ($n = 2$ and $M = 2$)

There are **collections of coefficients**

$$\left\{ C_{M,n}(\gamma) \right\}_{\gamma \in \{\boldsymbol{\gamma}^{(0,1)}, \boldsymbol{\gamma}^{(1,0)}, \boldsymbol{\gamma}^{(1,1)}\}}, \qquad \left\{ C_{\mathrm{B},M,n}(\gamma) \right\}_{\gamma \in \{\boldsymbol{\gamma}^{(0,1)}, \boldsymbol{\gamma}^{(1,0)}, \boldsymbol{\gamma}^{(1,1)}\}}$$

such that

$$\big(\mathrm{perm}(\boldsymbol{\theta})\big)^{M} = \sum_{\gamma \in \{\boldsymbol{\gamma}^{(0,1)}, \boldsymbol{\gamma}^{(1,0)}, \boldsymbol{\gamma}^{(1,1)}\}} C_{M,n}(\gamma) \cdot \boldsymbol{\theta}^{M \cdot \gamma},$$

$$\big(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big)^{M} = \sum_{\gamma \in \{\boldsymbol{\gamma}^{(0,1)}, \boldsymbol{\gamma}^{(1,0)}, \boldsymbol{\gamma}^{(1,1)}\}} C_{\mathrm{B},M,n}(\gamma) \cdot \boldsymbol{\theta}^{M \cdot \gamma}.$$

The following **bounds** hold

$$1 \leq \frac{C_{M,n}(\gamma)}{C_{\mathrm{B},M,n}(\gamma)} \leq 2, \qquad 1 \leq \frac{\big(\mathrm{perm}(\boldsymbol{\theta})\big)^{M}}{\big(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big)^{M}} < 2.$$

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example continued ($n = 2$ and **arbitrary** $M \in \mathbb{Z}_{\geq 1}$)

Generalizing the above result to the case where $n = 2$ and $M \in \mathbb{Z}_{\geq 1}$, the **coefficients** in $\big(\mathrm{perm}(\boldsymbol{\theta})\big)^{M}$ satisfy

$$C_{M,n}\big(\gamma^{(k,M-k)}\big) = \binom{M}{k}.$$

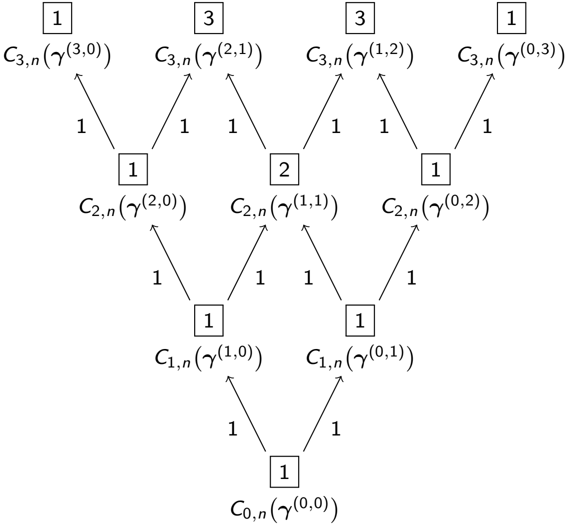
Note that the **recursion**

$$C_{M+1,n}\big(\gamma^{(k,M+1-k)}\big) = C_{M,n}\big(\gamma^{(k-1,M+1-k)}\big) + C_{M,n}\big(\gamma^{(k,M-k)}\big),$$

is **equivalent** to

$$\binom{M+1}{k} = \binom{M}{k-1} + \binom{M}{k}.$$

Analyzing $\mathrm{perm}(\theta)$ and $\mathrm{perm}_{\mathrm{B},M}(\theta)$



Pascal's triangle visualizing the **recursion** for $C_{M,n}$.

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

Example continued ($n = 2$ and **arbitrary** $M \in \mathbb{Z}_{\geq 1}$)

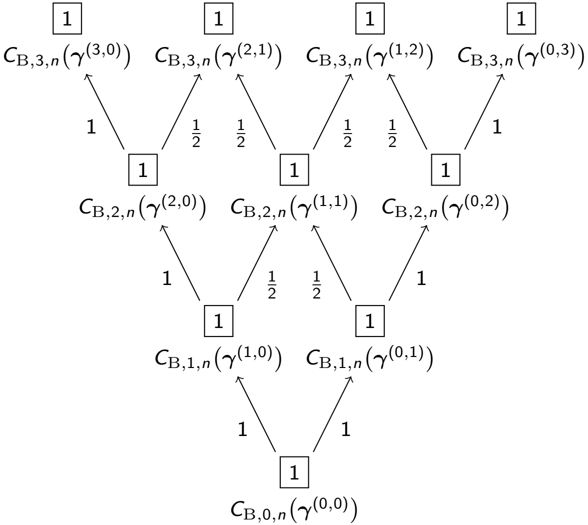
For the above **special setup**, the **coefficients** in $\left(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\right)^M$ satisfy

$$C_{\mathrm{B},M,n}\left(\gamma^{(k,M-k)}\right) = 1.$$

We have the **recursion**

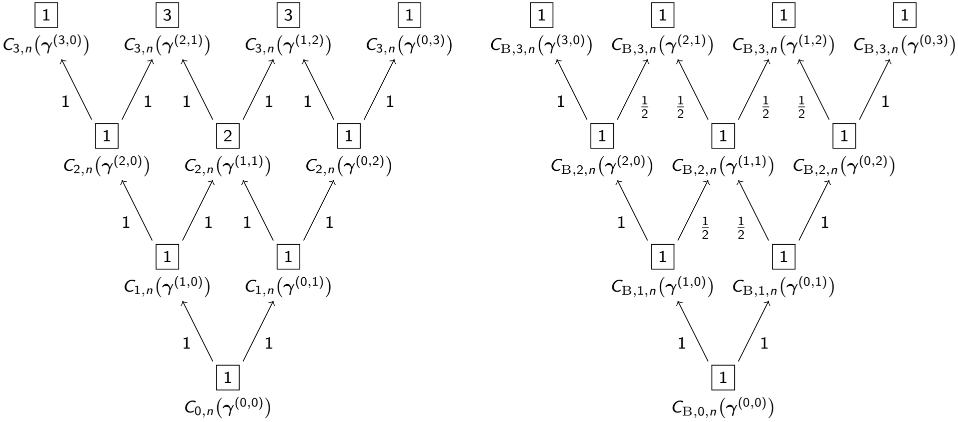
$$C_{\mathrm{B},M+1,n}\left(\gamma^{(k,M+1-k)}\right) = \begin{cases} C_{\mathrm{B},M,n}\left(\gamma^{(k,M-k)}\right) & k = 0 \\ C_{\mathrm{B},M,n}\left(\gamma^{(k-1,M+1-k)}\right) & k = M+1 \\ \frac{1}{2} \cdot C_{\mathrm{B},M,n}\left(\gamma^{(k-1,M+1-k)}\right) + \frac{1}{2} \cdot C_{\mathrm{B},M,n}\left(\gamma^{(k,M-k)}\right) & 1 \leq k \leq M \end{cases}$$

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$



Generalization of Pascal's triangle visualizing the recursion $C_{\mathrm{B},M,n}$.

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$



Visualizing the **recursions** of $C_{M,n}$ and $C_{\mathrm{B},M,n}$.

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

General Case (Arbitrary $n, M \in \mathbb{Z}_{\geq 1}$)

Lemma

Consider collections of **non-negative real numbers**

$$\big\{ C_{M,n}(\boldsymbol{\gamma}) \big\}_{\boldsymbol{\gamma} \in \Gamma_{M,n}}, \quad \big\{ C_{\mathrm{B},M,n}(\boldsymbol{\gamma}) \big\}_{\boldsymbol{\gamma} \in \Gamma_{M,n}}.$$

The **permanent** and its degree-M **Bethe permanent** satisfy

$$\big(\mathrm{perm}(\boldsymbol{\theta})\big)^{M} = \sum_{\boldsymbol{\gamma} \in \Gamma_{M,n}} \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}} \cdot C_{M,n}(\boldsymbol{\gamma}),$$

$$\Big(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\Big)^{M} = \sum_{\boldsymbol{\gamma} \in \Gamma_{M,n}} \boldsymbol{\theta}^{M \cdot \boldsymbol{\gamma}} \cdot C_{\mathrm{B},M,n}(\boldsymbol{\gamma}).$$

Analyzing $\mathrm{perm}(\boldsymbol{\theta})$ and $\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})$

General Case (**Arbitrary** $n, M \in \mathbb{Z}_{\geq 1}$)

Lemma

Let $M \in \mathbb{Z}_{\geq 2}$ and $\gamma \in \Gamma_{M,n}$. The following **recursions** hold

$$C_{M,n}(\gamma) = \sum_{\sigma_1 \in \mathcal{S}_{[n]}(\gamma)} C_{M-1,n}\big(\gamma_{\sigma_1}\big),$$

$$C_{\mathrm{B},M,n}(\gamma) = \frac{1}{\mathrm{perm}(\hat{\gamma}_{\mathcal{R},\mathcal{C}})} \cdot \sum_{\sigma_1 \in \mathcal{S}_{[n]}(\gamma)} C_{\mathrm{B},M-1,n}\big(\gamma_{\sigma_1}\big).$$

(The details of $\mathrm{perm}(\hat{\gamma}_{\mathcal{R},\mathcal{C}})$ and γ_{σ_1} are **omitted** here.)

Using **bounds** on $\mathrm{perm}(\hat{\gamma}_{\mathcal{R},\mathcal{C}})$ **proven** in

[Schrijver, 1998, Gurvits, 2011, Anari and Rezaei, 2019],

we can **bound** $C_{M,n}(\gamma)$ via $C_{\mathrm{B},M,n}(\gamma)$.

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Setup

A graphical-model-based approximation method

Finite graph covers

Analyzing the permanent and its degree-M Bethe permanent

▶ **Bounding the permanent via its approximations**

Conclusion

Bounding the permanent via its approximations

Lemma: We **bound** $C_{M,n}$ via $C_{\mathrm{B},M,n}$:

$$1 \leq \frac{C_{M,n}(\gamma)}{C_{\mathrm{B},M,n}(\gamma)} \leq (2^{n/2})^{M-1},$$

where the lower bound **resolves a conjecture** in

[Vontobel, 2013a].

Theorem: Based on

$$\big(\mathrm{perm}(\boldsymbol{\theta})\big)^{M} = \sum_{\gamma \in \Gamma_{M,n}} \boldsymbol{\theta}^{M \cdot \gamma} \cdot C_{M,n}(\gamma),$$

$$\big(\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})\big)^{M} = \sum_{\gamma \in \Gamma_{M,n}} \boldsymbol{\theta}^{M \cdot \gamma} \cdot C_{\mathrm{B},M,n}(\gamma),$$

we **bound** the permanent $\mathrm{perm}(\boldsymbol{\theta})$ via its **degree-M Bethe permanent**:

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})} < \left(2^{n/2}\right)^{\frac{M-1}{M}},$$

where the lower bound **resolves another conjecture** in

[Vontobel, 2013a].

Bounding the permanent via its approximations

We **bound** the permanent $\mathrm{perm}(\boldsymbol{\theta})$ via its degree-M **Bethe permanent**:

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})} < \left(2^{n/2}\right)^{\frac{M-1}{M}}.$$

Caveat: The proof uses the **bounds** in [Gurvits, 2011, Anari and Rezaei, 2019].

As $M \to \infty$,

$$1 \leq \liminf_{M\to\infty} \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B},M}(\boldsymbol{\theta})} \leq \lim_{M\to\infty} \left(2^{n/2}\right)^{\frac{M-1}{M}},$$

we **recover the bounds**

$$1 \leq \frac{\mathrm{perm}(\boldsymbol{\theta})}{\mathrm{perm}_{\mathrm{B}}(\boldsymbol{\theta})} \leq 2^{n/2}$$

where

- the **lower bound** proven in [Gurvits, 2011],
- the **upper bound** proven in [Anari and Rezaei, 2019].

Finite-graph-covers-based bounds for the permanent of a non-negative square matrix

Conclusion

- **Bound the matrix permanent** by the **degree-M Bethe permanents**.
- **Prove some of the conjectures** in [Vontobel, 2013a].
- Our proofs used some **rather strong results** from [Schrijver, 1998, Gurvits, 2011, Anari and Rezaei, 2019].

Open problems

- Find "**more basic**" proofs on the bounds.
- **Prove** the **recursions** for **general** S-FGs, *e.g.*, the S-FG defined based on **multi-affine homogeneous real stable** (MAHRS) polynomial.

Selected References I

Amit, A. and Linial, N. (2002).
Random graph coverings i: General theory and graph connectivity.
Combinatorica, 22(1):1–18.

Anari, N. and Gharan, S. O. (2021).
A generalization of permanent inequalities and applications in counting and optimization.
Adv. Math., 383:107657.

Anari, N. and Rezaei, A. (2019).
A tight analysis of Bethe approximation for permanent.
In *Proc. IEEE Ann. Symp. on Found. Comp. Sci.*, pages 1434–1445.

Selected References II

Angluin, D. (1980).

Local and global properties in networks of processors (extended abstract).

In *STOC*, STOC '80, page 82–93, New York, NY, USA. Association for Computing Machinery.

Borcea, J. and Brändén, P. (2009).

The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications.

Commun. Pure Appl. Math., 62(12):1595–1631.

Borcea, J., Brändén, P., and Liggett, T. M. (2009).

Negative dependence and the geometry of polynomials.

J. Amer. Math. Soc., 22(2):521–567.

Selected References III

Brändén, P. (2007).
Polynomials with the half-plane property and matroid theory.
Adv. Math., 216(1):302–320.

Brändén, P. (2014).
The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability.
Amer. J. Math., 136(1):241–253.

Brändén, P., Leake, J., and Pak, I. (2023).
Lower bounds for contingency tables via Lorentzian polynomials.
Israel J. Math., 253(1):43–90.

Selected References IV

Gharan, S. O. (2020).

Course notes of polynomial paradigm in algorithm design.

Lecture 3.

Gurvits, L. (2011).

Unleashing the power of Schrijver's permanental inequality with the help of the Bethe approximation.

Elec. Coll. Comp. Compl.

Gurvits, L. (2015).

Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: Combinatorial and algorithmic applications.

Inform. and Comput., 240:42–55.

Selected References V

Koetter, R. and Vontobel, P. O. (2003).

Graph covers and iterative decoding of finite-length codes.

In *ISTC*, pages 75–82, Brest, France.

Marcus, A. W., Spielman, D. A., and Srivastava, N. (2015).

Interlacing families i: Bipartite ramanujan graphs of all degrees.

Ann. of Math., 182:307–325.

Oxley, J. (2011).

Matroid Theory.

Oxford University Press, New York, NY, USA, 2nd edition.

Schrijver, A. (1998).

Counting 1-factors in regular bipartite graphs.

J. Comb. Theory, Ser. B, 72(1):122–135.

Selected References VI

Straszak, D. and Vishnoi, N. K. (2017).
Real stable polynomials and matroids: optimization and counting.
In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2017, page 370–383, New York, NY, USA. Association for Computing Machinery.

Straszak, D. and Vishnoi, N. K. (2019).
Belief propagation, Bethe approximation, and polynomials.
IEEE Trans. Inf. Theory, 65(7):4353–4363.

Vontobel, P. O. (2013a).
The Bethe permanent of a nonnegative matrix.
IEEE Trans. Inf. Theory, 59(3):1866–1901.

Selected References VII

Vontobel, P. O. (2013b).

Counting in graph covers: A combinatorial characterization of the Bethe entropy function.

IEEE Trans. Inf. Theory, 59(9):6018–6048.

Wainwright, M. J. and Jordan, M. I. (2008).

Graphical models, exponential families, and variational inference.

Foundation and Trends in Machine Learning, 1(1–2):1–305.

Thank you!

Y. H. gratefully acknowledges useful discussions with Prof. Jonathan Leake.