Sets of Marginals and Pearson-Correlation-based CHSH Inequalities for a Two-Qubit System

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Overview

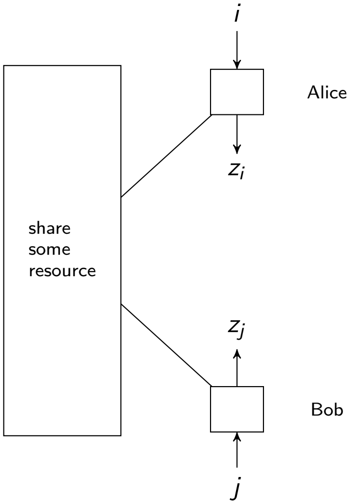
1. Analysis of the **EPR** experiment in terms of normal factor graphs (NFGs) of simple quantum mass functions (SQMFs).

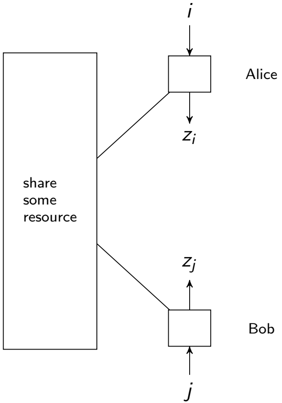
2. A New **CHSH** style inequality.

The EPR Experiment

The EPR Experiment

- Alice and Bob shared resource.
- The control value on Alice side is $i \in \{1, 3\}$.
- The control value on Bob side is $j \in \{2, 4\}$.
- The measurement outcome on Alice side is $z_i \in \{-1, 1\}$.
- The measurement outcome on Bob side is $z_j \in \{-1, 1\}$.





If the shared common resource is **classical**, then the **random variables** $Z_1, \ldots, Z_4 \in \{-1, 1\}$ with realizations $z_1, \ldots, z_4 \in \{-1, 1\}$ satisfy the **CHSH inequality**:

$$\left|\mathbb{E}(Z_1 \cdot Z_2) + \mathbb{E}(Z_1 \cdot Z_4) + \mathbb{E}(Z_3 \cdot Z_2) - \mathbb{E}(Z_3 \cdot Z_4)\right| \leq 2.$$

The CHSH Inequality

For random variables $Z_1, \ldots, Z_4 \in \{-1, 1\}$, we have

$$\left|\mathbb{E}(Z_1 \cdot Z_2) + \mathbb{E}(Z_1 \cdot Z_4) + \mathbb{E}(Z_3 \cdot Z_2) - \mathbb{E}(Z_3 \cdot Z_4)\right| \leq 2.$$

Proof.

It holds that

$$Z_1 \cdot Z_2 + Z_1 \cdot Z_4 + Z_3 \cdot Z_2 - Z_3 \cdot Z_4 = Z_1 \cdot (Z_2 + Z_4) + Z_3 \cdot (Z_2 - Z_4).$$

1. If $Z_2 = Z_4$, we have

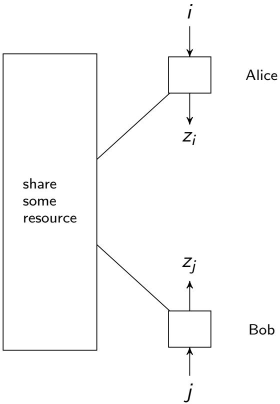
$$Z_1 \cdot (Z_2 + Z_4) = \begin{cases} 2Z_1 & Z_2 = 1 \\ -2Z_1 & Z_2 = -1 \end{cases}$$

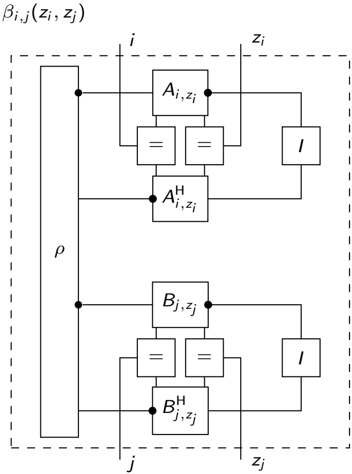
2. If $Z_2 \neq Z_4$, we have

$$Z_3 \cdot (Z_2 - Z_4) = \begin{cases} 2Z_3 & Z_2 = 1 \\ -2Z_3 & Z_2 = -1 \end{cases}$$

∎

The EPR Experiment

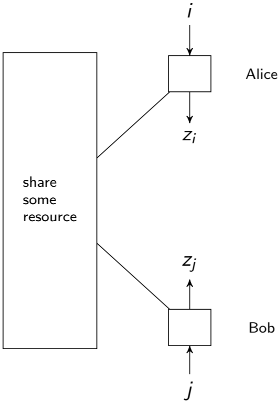


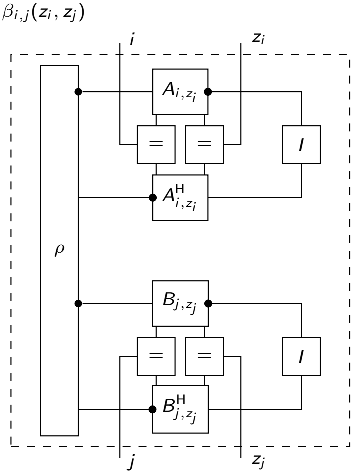


If the shared common resource is an **entangled quantum system**, then $Z_1, \ldots, Z_4 \in \{-1, 1\}$ are **observables**. The average values for the observable $Z_i \cdot Z_j$ is

$$\langle Z_i \cdot Z_j \rangle = \sum_{z_i, z_j} z_i \cdot z_j \cdot \beta_{i,j}(z_i, z_j), \qquad \{i, j\} \in \big\{\{1,4\}, \{1,2\}, \{3,4\}, \{3,2\}\big\}.$$

The EPR Experiment





There exist ρ, $\{A_{i,z_i}\}_{i,z_i}$, and $\{B_{j,z_j}\}_{j,z_j}$ s.t.

$$\langle Z_1 \cdot Z_2 \rangle + \langle Z_1 \cdot Z_4 \rangle + \langle Z_3 \cdot Z_2 \rangle - \langle Z_3 \cdot Z_4 \rangle = 2\sqrt{2}.$$

New CHSH style inequality

New CHSH style inequality

Whether **other measures of correlations** can be used for devising CHSH inequality.

Consider the case where the **full data are not available**, but only certain specific (not necessarily linear) functions of the joint probabilities are.

For random variables $Z_1, \ldots, Z_4 \in \{-1, 1\}$, the authors in [Pozsgay et al., 2017] proved that

$$\left|\mathrm{Cov}(Z_1, Z_2) + \mathrm{Cov}(Z_1, Z_4) + \mathrm{Cov}(Z_3, Z_2) - \mathrm{Cov}(Z_3, Z_4)\right| \leq \frac{16}{7},$$

where $\mathrm{Cov}(Z_i, Z_j)$ is the **covariance** of random variables Z_i and Z_j.

They also conjectured that

$$\left|\mathrm{Corr}(Z_1, Z_2) + \mathrm{Corr}(Z_1, Z_4) + \mathrm{Corr}(Z_3, Z_2) - \mathrm{Corr}(Z_3, Z_4)\right| \leq \frac{5}{2}.$$

New CHSH style inequality

Theorem

Suppose that the random variables $Z_1, \ldots, Z_4 \in \{-1, 1\}$ *satisfy*

$$\mathrm{Var}(Z_1),\ \ldots,\ \mathrm{Var}(Z_4) \in \mathbb{R}_{>0}.$$

The **Pearson correlation coefficient (PCC)-based CHSH inequality** *holds:*

$$\left|\mathrm{Corr}(Z_1, Z_2) + \mathrm{Corr}(Z_1, Z_4) + \mathrm{Corr}(Z_3, Z_2) - \mathrm{Corr}(Z_3, Z_4)\right| \leq \frac{5}{2}.$$

which resolves a conjecture proposed in [Pozsgay et al., 2017].

Proof.

See [Huang and Vontobel, 2021]. ■

Remark

This inequality is a **non-linear** *function w.r.t. the probability of the random variables* $Z_1, \ldots, Z_4 \in \{-1, 1\}$.

Outline

- Introduction
  - Standard Normal Factor Graphs (S-NFG) and its associated Probability Mass Functions (PMFs)
  - Quantum Mass Functions (QMFs)
  - Simple Quantum Mass Functions (SQMFs)
  - An example SQMF for the EPR Experiment
- Main Results
  - Sets of Marginals for Example PMFs and SQMFs

Introduction

Standard Normal Factor graphs (S-NFGs)

- Introduction of Normal Factor Graphs (NFGs)
- Definition of S-NFGs
- PMFs induced by an S-NFG

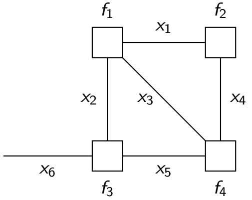
Introduction of Normal Factor Graphs (NFGs)

Introduction of NFGs

- NFG are used to represent **factorizations of multivariate functions**.
- Many inference problems can be formulated as **computing the marginals of some multivariate functions**.
- The word "normal" refers to the fact that variables are arguments of only one or two local functions.

Example

$$g(x_1,\ldots,x_4) \triangleq f_1(x_1,x_2,x_3)\cdot f_2(x_1,x_4) \cdot f_3(x_2,x_5,x_6)\cdot f_4(x_3,x_4,x_5)$$



Consider a factor graph.

- A half edge: an edge incident on **one** function node
- A full edge: an edge incident on **two** function nodes.

Definition of S-NFGs

Definition of S-NFGs

Definition

The S-NFG $\mathsf{N}(\mathcal{F}, \mathcal{E}, \mathcal{X})$ consists of:

1. **the graph** $(\mathcal{F}, \mathcal{E})$ with vertex set $\mathcal{F}$ and edge set $\mathcal{E}$, where
   - $\mathcal{E}$ consists of all full edges and half edges in N,
   - $\mathcal{F}$ is the set of function nodes;
2. **the alphabet** $\mathcal{X} := \prod_{e \in \mathcal{E}} \mathcal{X}_e$, where $\mathcal{X}_e$ is the alphabet associated with edge $e \in \mathcal{E}$.

An $f \in \mathcal{F}$ will denote a function node and the corresponding local function.

Definition of S-NFGs

Definition

Let $\langle \mathcal{R}, +, \cdot \rangle$ be a **ring**. Given $\mathsf{N}(\mathcal{F}, \mathcal{E}, \mathcal{X})$, we make the following definitions.

1. The **local function** f associated with function node $f \in \mathcal{F}$ denotes an arbitrary mapping
$$f : \prod_{e \in \partial f} \mathcal{X}_e \to \mathcal{R}.$$

2. The **global function** is defined to be
$$g(\boldsymbol{x}) \triangleq \prod_{f \in \mathcal{F}} f(\boldsymbol{x}_{\partial f}).$$

3. The **partition function** is defined to be
$$Z(\mathsf{N}) \triangleq \sum_{\boldsymbol{x}} g(\boldsymbol{x}).$$

Definition of S-NFGs

Definition

If the ring $\mathcal{R}$ in the definition of local functions is the set of nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0}$, then we make further definitions.

4. The **probability mass function (PMF)** induced on N is defined to be the function

$$p(\boldsymbol{x}) \triangleq g(\boldsymbol{x})/Z(\mathsf{N}).$$

5. Let $\mathcal{I}$ be a **subset** of $\mathcal{E}$ and let $\mathcal{I}^{\mathrm{c}} \triangleq \mathcal{E} \setminus \mathcal{I}$ be its complement. The **marginal** $p_{\mathcal{I}}$ is defined to be

$$p_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}) \triangleq \sum_{\boldsymbol{x}_{\mathcal{I}^{\mathrm{c}}}} p(\boldsymbol{x}), \qquad \boldsymbol{x}_{\mathcal{I}} \in \mathcal{X}_e^{|\mathcal{I}|}.$$

In case of setups with multiple NFGs, we add index N in functions g and p. For general definition, we simply omit this index.

There is no loss of generality of the case where the number of edges incident on each function node is two.

Definition of S-NFGs

1. Local functions: $f_{1,4}, \ldots, f_{3,4}$;
2. Set of edges: $\mathcal{E}_{\text{full}} = \{1, 2, 3, 4\}$;
3. Global function:

$$g_{\mathsf{N}_1}(x_1, \ldots, x_4) = f_{1,4}(x_1, x_4) \cdot f_{1,2}(x_1, x_2) \cdot f_{3,4}(x_3, x_4) \cdot f_{3,2}(x_3, x_2);$$

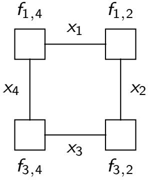
4. Partition function: $Z(\mathsf{N}_1) = \sum_{x_1,\ldots,x_4} g_{\mathsf{N}_1}(x_1, \ldots, x_4)$.
5. Probability mass function:

$$p_{\mathsf{N}_1}(x_1, \ldots, x_4) = g_{\mathsf{N}_1}(x_1, \ldots, x_4)/Z(\mathsf{N})$$

6. Let the set $\mathcal{I}$ be $\mathcal{I} \subseteq \{1, 2, 3, 4\}$.
7. The marginals:

$$p_{\mathsf{N}_1,\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}) = \sum_{\boldsymbol{x}_{\mathcal{I}^{\mathrm{c}}}} p_{\mathsf{N}_1}(\boldsymbol{x}), \quad \boldsymbol{x}_{\mathcal{I}} \in \mathcal{X}_e^{|\mathcal{I}|},$$

$$p_{\mathsf{N}_1,\{i,j\}}(x_i, x_j) = \sum_{\boldsymbol{x}_{\{1,2,3,4\}\setminus\{i,j\}}} p_{\mathsf{N}_1}(x_1, \ldots, x_4).$$



The S-NFG N_1.

PMFs induced by an S-NFG

PMFs induced by an S-NFG

Consider a sequence $Y_1, \ldots, Y_n$ of random variables with the joint PMF

$$P_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n), \qquad y_1 \in \mathcal{Y}_1, \ldots, y_n \in \mathcal{Y}_n.$$

In a typical scenario of interest, we might have observed

$$Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}$$

and would like to estimate Y_n based on these observations.

Usually, $P_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$ does not have a "nice" factorization.

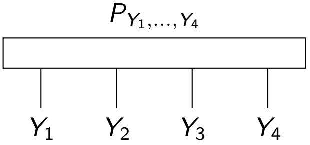
However, very often it is possible to find a function $p(\boldsymbol{x}, \boldsymbol{y})$ such that

1. $p(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_{\geq 0}$ for all $\boldsymbol{x}, \boldsymbol{y}$;
2. $\sum_{\boldsymbol{x},\boldsymbol{y}} p(\boldsymbol{x}, \boldsymbol{y}) = 1$;
3. $\sum_{\boldsymbol{x}} p(\boldsymbol{x}, \boldsymbol{y}) = P_{\boldsymbol{Y}}(\boldsymbol{y})$ for all $\boldsymbol{y}$;
4. $p(\boldsymbol{x}, \boldsymbol{y})$ has a "nice" factorization.

Note that $p(\boldsymbol{x}, \boldsymbol{y})$ represents a joint PMF over $\boldsymbol{x}$ and $\boldsymbol{y}$.

PMFs induced by an S-NFG

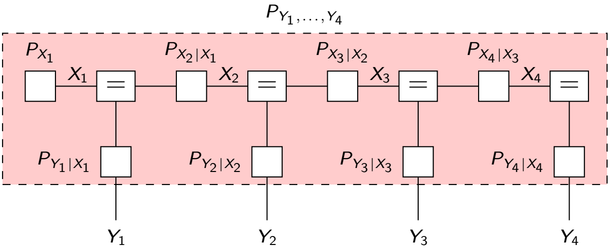
Example (A Hidden Markov Model)



$$P_{Y_1,\ldots,Y_4}(y_1,\ldots,y_4).$$

PMFs induced by an S-NFG

Example (A Hidden Markov Model)



$$p(x_1,\ldots,x_4,y_1,\ldots,y_4) = P_{X_1}(x_1)\cdot\left(\prod_{i=1}^{3} P_{X_{i+1}|X_i}(x_{i+1}|x_i)\right)\cdot\left(\prod_{i=1}^{4} P_{Y_i|X_i}(y_i|x_i)\right),$$

$$P_{Y_1,\ldots,Y_4}(y_1,\ldots,y_4) = \sum_{x_1,\ldots,x_4} p(x_1,\ldots,x_4,y_1,\ldots,y_4)$$

After applying a closing-the-box (CTB) operation to the above factor graph, i.e., summing over the variables associated with the internal edges, we obtain $P_{Y_1,\ldots,Y_4}$.

Quantum Mass Functions (QMFs)

- Definition of QMFs
- An Example Factor Graph for a Quantum Information Process
- Definition Simple Quantum Mass Functions (SQMFs)
- An SQMF for the EPR Experiment

Definition of QMFs

Definition of QMFs

Consider again a sequence of random variables $Y_1, \ldots, Y_n$ with the joint PMF

$$P_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n), \qquad y_1 \in \mathcal{Y}_1, \ldots, y_n \in \mathcal{Y}_n.$$

However, now we assume that these random variables represent the measurements obtained by running some quantum-mechanical experiment.

Again, a typical scenario of interest is that we would like to estimate variable Y_n based on the observations

$$Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}.$$

QMFs

In general, the PMF $P_{\boldsymbol{Y}}(\boldsymbol{y})$ does not have a "nice" factorization.

However, frequently it is possible to introduce suitable auxiliary quantum variables $x_1, \ldots x_m, x_1', \ldots, x_m'$ such that there is a function $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ satisfying

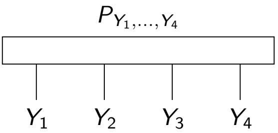
1. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) \in \mathbb{C}$ for all $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}$;
2. $\sum_{\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}} q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = 1$;
3. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ is a positive semi-definite (PSD) kernel in quantum variables $(\boldsymbol{x}, \boldsymbol{x}')$ for every classical variable $\boldsymbol{y}$;
4. $\sum_{\boldsymbol{x}, \boldsymbol{x}'} q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}) = P_{\boldsymbol{Y}}(\boldsymbol{y})$;
5. $q(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y})$ has a "nice" factorization.

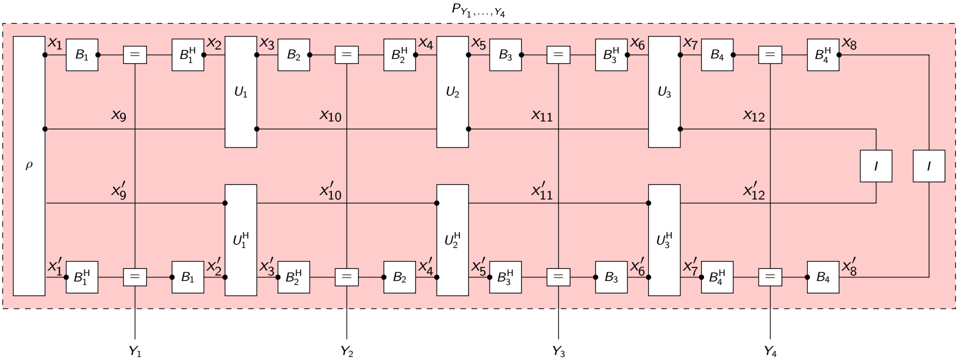
The function q is called a QMF in [Loeliger and Vontobel, 2017].

An Example Factor Graph for a QIP

An Example Factor Graph for a QIP

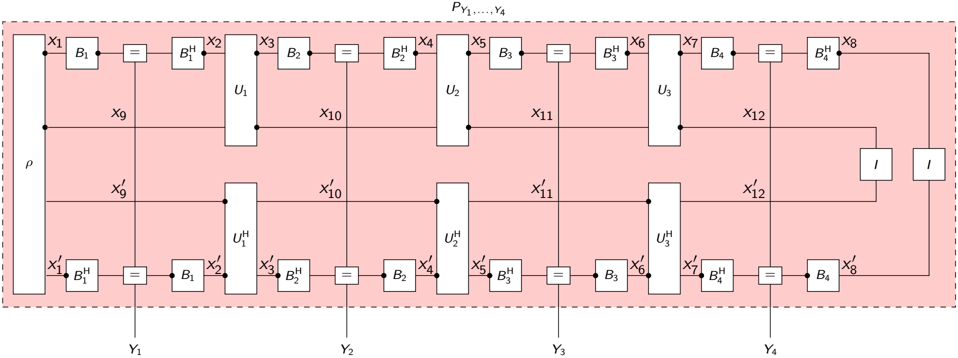
Example





An Example Factor Graph for a QIP

Example



- A system prepared in ρ.
- Partial measurements $B_1, \ldots, B_4$.
- Unitary evolutions $U_1, \ldots, U_3$.
- Quantum variables $(x_1, \ldots, x_{12})$ and $(x'_1, \ldots, x'_{12})$.

After applying a CTB operation to the above factor graph, we obtain $P_{Y_1,\ldots,Y_4}$.

Definition of SQMFs

Definition of SQMFs

Interesting enough, it is sufficient to consider the SQMF where the classical variable $\boldsymbol{y}$ in QMF do not appear explicitly in SQMF anymore. However, as we will see later, classical variable $\boldsymbol{y}$ emerges from SQMFs.

Definition

An SQMF $q(\boldsymbol{x}, \boldsymbol{x}')$ satisfies

1. $q(\boldsymbol{x}, \boldsymbol{x}') \in \mathbb{C}$ for all $\boldsymbol{x}, \boldsymbol{x}'$;
2. $\sum_{\boldsymbol{x}, \boldsymbol{x}'} q(\boldsymbol{x}, \boldsymbol{x}') = 1$;
3. $q(\boldsymbol{x}, \boldsymbol{x}')$ is a PSD kernel in $(\boldsymbol{x}, \boldsymbol{x}')$.

Definition of SQMFs

Definition

For $\boldsymbol{x} = (x_1, \ldots, x_m)$, let $\mathcal{I} \subseteq \{1, \ldots, m\}$ and let $\mathcal{I}^{\mathrm{c}}$ be its complement. The variable $\boldsymbol{x}_{\mathcal{I}}$ is defined to be $\boldsymbol{x}_{\mathcal{I}} = (x_k)_{k \in \mathcal{I}}$.

The variables in $\boldsymbol{x}_{\mathcal{I}}$ are called jointly **classicable** if the marginalized SQMF

$$q_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}'_{\mathcal{I}}) \triangleq \sum_{\boldsymbol{x}_{\mathcal{I}^{\mathrm{c}}}, \boldsymbol{x}'_{\mathcal{I}^{\mathrm{c}}}} q(\boldsymbol{x}, \boldsymbol{x}')$$

is zero for all $(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}'_{\mathcal{I}})$ satisfying $\boldsymbol{x}_{\mathcal{I}} \neq \boldsymbol{x}'_{\mathcal{I}}$.

Definition

If the variables in $\boldsymbol{x}_{\mathcal{I}}$ are jointly classicable then

$$p(\boldsymbol{x}_{\mathcal{I}}) \triangleq q_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}}, \boldsymbol{x}_{\mathcal{I}}), \qquad \boldsymbol{x}_{\mathcal{I}} \in \mathcal{X}_{\mathcal{I}},$$

represents a joint PMF over $\boldsymbol{x}_{\mathcal{I}}$.

Properties of SQMFs

Remark

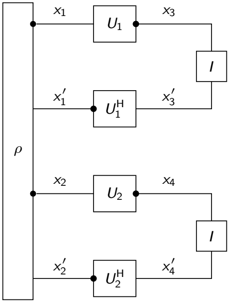
- *By defining $p(\mathbf{x}_{\mathcal{I}}) \triangleq q_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}, \mathbf{x}_{\mathcal{I}})$, we can see that classical variable $\mathbf{y}$ that were omitted when going from QMFs to SQMFs can "emerge" again.*
- *Note that there is a strong connection of SQMFs to the so-called decoherence functional [Gell-Mann and Hartle, 1989, Dowker and Halliwell, 1992], and via this also to the consistent-histories approach to quantum mechanics [Griffiths, 2002]. However, the starting point of our investigations is quite different.*

An SQMF for the EPR Experiment

An SQMF for the EPR Experiment

The PMF for the measurement outcomes in the EPR experiment can be obtained by the marginals of the SQMF represented by the following NFG.

- Variables x_1 and x_2 are jointly classicable variables, i.e., the marginal $q_{1,2}(x_1, x_2, x_1', x_2') = 0$ when $x_1 \neq x_1'$ or $x_2 \neq x_2'$;
- Variables x_1 and x_4 are jointly classicable variables;
- Variables x_3 and x_2 are jointly classicable variables;
- Variables x_3 and x_4 are jointly classicable variables;
- However, variables $x_1, \ldots, x_4$ are not jointly classicable variables, i.e, exists $\boldsymbol{x}, \boldsymbol{x}'$ s.t. $q(\boldsymbol{x}, \boldsymbol{x}') \neq 0$ when $\boldsymbol{x} \neq \boldsymbol{x}'$;



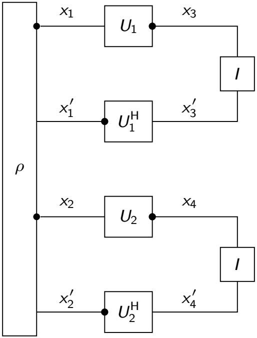
An SQMF for the EPR Experiment

Example

Consider the following NFG, where

$$\mathcal{X}_i = \mathcal{X}_i' \triangleq \{0, 1\}, \qquad i \in \{1, \ldots, 4\}, \qquad U_1 = U_2 \triangleq \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\boldsymbol{\psi} \triangleq \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}^{\mathsf{T}}, \qquad \rho \triangleq \boldsymbol{\psi} \cdot \boldsymbol{\psi}^{\mathsf{H}}.$$



An SQMF for the EPR Experiment

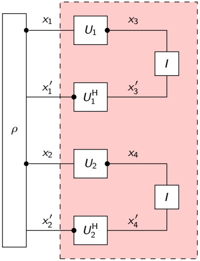
The following matrix shows the components of the SQMF $q(\boldsymbol{x}, \boldsymbol{x}')$, where both the row index $(x_1, \ldots, x_4)$ and column index $(x_1', \ldots, x_4')$ range over $(0,0,0,0), (0,0,0,1), \ldots, (1,1,1,1)$.

$$
\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
\alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

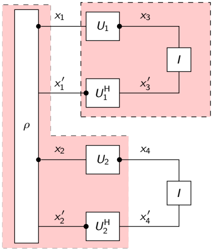
Here: $\alpha_1 \triangleq 0.0833$, $\beta_1 \triangleq -0.0833$.

Note that the above matrix is not diagonal.

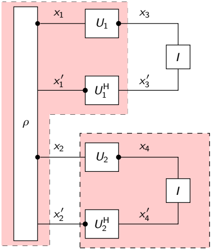
An SQMF for EPR Experiment



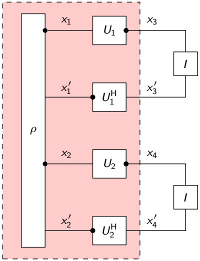
$$q_{1,2} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$q_{1,4} = \frac{1}{6}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

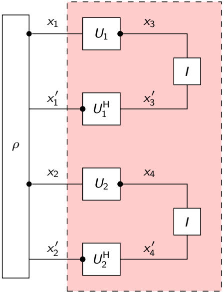


$$q_{3,2} = \frac{1}{6}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

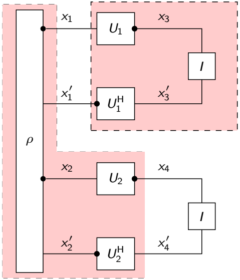


$$q_{3,4} = \frac{1}{12}\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

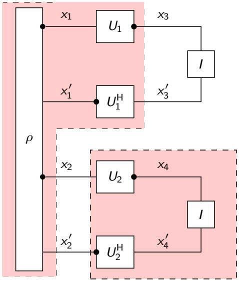
An SQMF for the EPR Experiment



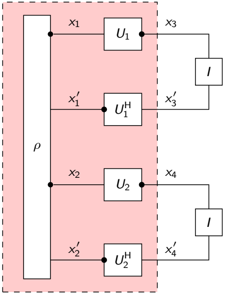
$$p_{1,2} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$



$$p_{1,4} = \frac{1}{6}\begin{pmatrix} 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$



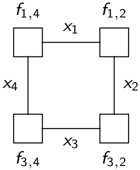
$$p_{3,2} = \frac{1}{6}\begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$



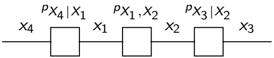
$$p_{3,4} = \frac{1}{12}\begin{pmatrix} 9 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Main Results

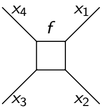
Main Results



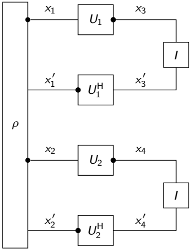
The S-NFG N_1.



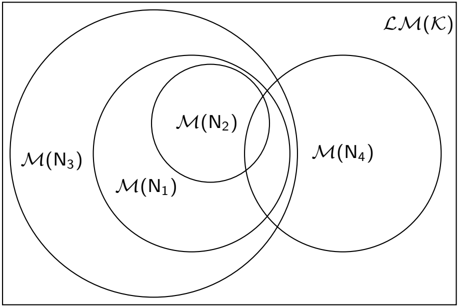
The S-NFG N_2.



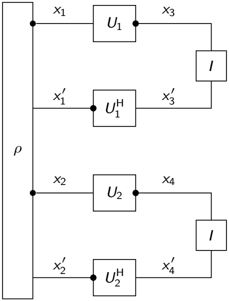
The S-NFG N_3.



The NFG N_4.



Conclusion



Proposition

- *We characterize the relationships among the sets of marginals mentioned in the previous slides.*
- *Many well-known quantum phenomena, e.g., Hardy's paradox and Bell's test, can be cast with this SQMF.*

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Thank you!

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Backup Slides

Combining Implications Obtained by the Marginals

Combining Implications

1. The marginal $p_{3,4}(1,1) = \frac{1}{12}$ shows that it is possible to have $x_3 = x_4 = 1$.

2. The marginals $p_{3,2}(1,0) = 0$ and $p_{3,2}(1,1) = 1/6$ show that the condition $x_3 = 1$ implies $x_2 = 1$.

3. The marginals $p_{1,4}(0,1) = 0$ and $p_{1,4}(1,1) = 1/6$ show that the condition $x_4 = 1$ implies $x_1 = 1$.

4. However, the marginal $p_{1,2}(1,1) = 0$ implies that we cannot have $x_1 = x_2 = 1$, which contradicts $p_{3,4}(1,1) > 0$.

$$S_1 : \; x_3 = x_4 = 1 \quad \Longrightarrow \quad S_2 : \; x_2 = 1$$

$$S_1 : \; x_3 = x_4 = 1 \quad \Longrightarrow \quad S_3 : \; x_1 = 1$$

$$S_1 : \; x_3 = x_4 = 1 \quad \Longrightarrow \quad S_4 : \; x_1 = x_2 = 1$$

Remarks on the SQMF for the EPR Experiment

Remark

- *Typically, the set of marginals* $\{p_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}})\}_{\mathcal{I} \in \mathcal{K}}$ *is "incompatible", i.e., there is no PMF* $p(\boldsymbol{x})$ *such that* $p_{\mathcal{I}}(\boldsymbol{x}_{\mathcal{I}})$ *is a marginal of* $p(\boldsymbol{x})$ *for all* $\mathcal{I} \in \mathcal{K}$.

- *Other paradoxes (e.g. Bell's test, Wigner's friend experiment, and the Frauchiger-Renner paradox) can also be expressed in terms of some suitably defined SQMFs.*

Main Results

Definition

$$\mathrm{Corr}(\boldsymbol{\beta}_{i,j}) \triangleq \frac{\beta_{i,j}(0,0)\cdot\beta_{i,j}(1,1)-\beta_{i,j}(0,1)\cdot\beta_{i,j}(1,0)}{\sqrt{\beta_i(0)\cdot\beta_i(1)\cdot\beta_j(0)\cdot\beta_j(1)}},$$

$$\mathrm{CorrCHSH}(\boldsymbol{\beta}) \triangleq \mathrm{Corr}(\boldsymbol{\beta}_{1,2})+\mathrm{Corr}(\boldsymbol{\beta}_{1,4})+\mathrm{Corr}(\boldsymbol{\beta}_{3,2})-\mathrm{Corr}(\boldsymbol{\beta}_{3,4}),$$

$$\mathcal{LM}_{\mathrm{CHSH}}(\mathcal{K}) \triangleq \{\boldsymbol{\beta}\in\mathcal{LM}(\mathcal{K}) \mid \text{(1) and (2) hold}\},$$

where

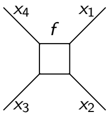
$$\beta_i(0)\cdot\beta_i(1) > 0, \qquad i\in\mathcal{E}(\mathsf{N}_1), \tag{1}$$

$$\sum_{\{i,j\}\in\mathcal{K}} (-1)^{[i=3,j=4]}\cdot\Big(\beta_{i,j}(0,0)+\beta_{i,j}(1,1)-\beta_{i,j}(0,1)-\beta_{i,j}(1,0)\Big) \leq 2, \tag{2}$$

Inequalities in (2) are inspired by the CHSH inequality. We prove this inequality by showing

$$\sup_{\boldsymbol{\beta}\in\mathcal{LM}_{\mathrm{CHSH}}(\mathcal{K})} \mathrm{CorrCHSH}(\boldsymbol{\beta}) = \frac{5}{2}.$$

Main Results



S-NFG N_3.

$$\sup_{\boldsymbol{\beta} \in \mathcal{LM}_{\mathrm{CHSH}}(\mathcal{K})} \mathrm{CorrCHSH}(\boldsymbol{\beta}) = \frac{5}{2}.$$

Suppose that we consider

$$\max_{\boldsymbol{\beta} \in \mathcal{M}(\mathsf{N}_3)} \mathrm{CorrCHSH}(\boldsymbol{\beta}).$$

For any $\boldsymbol{\beta} \in \mathcal{M}(\mathsf{N}_3)$, the marginal $\boldsymbol{\beta}_{i,j}$ can be written as a convex combination of some joint PMF for $X_1, \ldots, X_4$, i.e., $\{p_{\mathsf{N}_3}(\boldsymbol{x})\}_{\boldsymbol{x}}$, which makes the expression of $\mathrm{CorrCHSH}(\boldsymbol{\beta})$ non-trivial. By considering a superset of $\mathcal{M}(\mathsf{N}_3)$, i.e., $\mathcal{LM}_{\mathrm{CHSH}}(\mathcal{K})$, we can simplify $\mathrm{CorrCHSH}(\boldsymbol{\beta})$.