On constrained coding and contingency tables

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Southeast University

Today's talk is based on a **joint work with Professor Pascal O. Vontobel** at CUHK:

Y. Huang, P. O. Vontobel, "The Bethe partition function and the SPA for factor graphs based on homogeneous real stable polynomials,"
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3029-3034.

Outline

Motivation

A setup based on binary matrices with prescribed row sums and column sums

Graphical-model-based approximation method

Main results

A more general setup

Main results for a more general setup

Numerical results

Future works and connection to other works

Outline

- **Motivation**

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Contingency Table

A contingency table is a type of **table** in a **matrix format** that displays the **multivariate frequency distribution** of the variables.

Example of a 2×2 contingency table

Suppose we have two **categorical variables**:

1. **gender** (male or female),
2. **handedness** (right or left handed).

Conduct a **simple random sampling** and obtain a **size 100 data** that is **summarized** by the **following table**:

	Right-hand	Left-hand	Total
Male	50	6	56
Female	40	4	44
Total	90	10	100

Contingency Table

Contingency tables are **fundamental** objects across the sciences including

1. **statistics** [FLL17][1],
2. **combinatorics and graph theory** [Bar09][2],
3. **discrete geometry and combinatorial optimization** [DK14][3],
4. **algebraic and enumerative combinatorics** [PP20][4].

[1] M. W. Fagerland, S. Lydersen and P. Laake, Statistical analysis of contingency tables, CRC Press, 2017.

[2] A. Barvinok, "Asymptotic estimates for the number of contingency tables, integer flows, and volumes of transportation polytopes," Int. Math. Res. Notices, 2009.

[3] J. A. De Loera and E. D. Kim, "Combinatorics and geometry of transportation polytopes: an update," Discrete geometry and algebraic combinatorics, 2014

[4] I. Pak and G. "Panova, Bounds on Kronecker coefficients via contingency tables", Linear Algebra Appl., 2020.

Two-dimensional constrained coding

Why do we need **constrained coding**?

Answer from [MMH01][5]:

- In most **data recording** systems and many **data communication systems**, some sequences are **more prone to error** than others.
- In order to **reduce the likelihood of error**, we need to impose **constraints on the sequences** that are allowed to be **recorded or transmitted**.
- Given such constraints, it is then necessary to **encode arbitrary user sequences** into sequences that **satisfy the constraint**.

[5] B. H. Marcus, R. M. Roth, and P H. Siegel, "An introduction to coding for constrained systems," Lecture notes, 2001.

Two-dimensional constrained coding

Why do we consider **two-dimensional constrained coding**?

Answer from [N+23][6]:

1. In **optical recording**, *e.g.*, recording data in a **CD**, the recording device is a **surface**, and the recording data is in **two dimensional**.
2. In **resistive memory technologies**, the memory cell is a passive **two-terminal device** that can be **both read and written** over a simple crossbar structure.
3. These two models offer a **huge density advantage**, however, face new reliability issues and introduce **two-dimensional constraints**.

[6] T. T. Nguyen, K. Cai, H. M. Kiah, K. A. S. Immink, and Y. M. Chee, "Two-dimensional RC/SW constrained codes bounded weight and almost balanced weight," IEEE Trans. Inf. Theory, 2023.

Two-dimensional constrained coding

In this talk, we discuss **two-dimensional constrained binary coding** where the constrains are specified by **the column sum and the row sums**.

An introductory example

Consider the set of all **binary** 3×3 **matrices**.

We want to know the number of **binary** 3×3 matrices with **row sums** and **column sums** equaling **two**.

The following are **example binary** 3×3 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Two-dimensional constrained coding

An introductory example

Consider the set of all **binary** 3×3 **matrices**.

We want to know the number of **binary** 3×3 matrices with **row sums** and **column sums** equaling **two**.

The following are **example binary** 3×3 matrices:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{\times}, \quad \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}}_{\checkmark}, \quad \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{\checkmark}.$$

The number of such matrices is **3!**.

Two-dimensional constrained coding

An introductory example

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

- These binary matrices can be viewed as **binary contingency tables** of size 3×3 with **row sums** and **column sums** equaling **two**.
- The number of such **binary contingency tables** is $3!$.

We focus on **counting the number** of **binary contingency tables** / **two-dimensional binary codes** with prescribed row sums and column sums.

This counting problem is **non-trivial** in general.

Two-dimensional constrained binary codes and contingency tables

A great deal of effort is made to **approximate** the number of **contingency tables with prescribed row sums and column sums**.

A **variety of tools** have been developed in **different areas**, such as

1. the **traditional and probabilistic divide-and-conquer** [DZ15+][7],
2. the **Markov chain Monte Carlo (MCMC)** algorithms [C+06][8],
3. **approximation algorithms** [B+10][9].

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[7] S. DeSalvo and J. Y. Zhao, "Random sampling of contingency tables via probabilistic divide-and-conquer," Comput. Statist., 2020.

[8] M. Cryan, M. Dyer, L. A. Goldberg, M. Jerrum and R. Martin, "Rapidly mixing Markov chains for sampling contingency tables with a constant number of rows," SIAM J. Comput., 2006.

[9] A. Barvinok, Z. Luria, A. Samorodnitsky and A. Yong, "An approximation algorithm for counting contingency tables," Random Structures Algorithms, 2010

Outline

Setup

Definition

1. $[n] \triangleq \{1, 2, \ldots, n\}$ for $n \in \mathbb{Z}_{\geq 1}$ and $[m] \triangleq \{1, 2, \ldots, m\}$ for $m \in \mathbb{Z}_{\geq 1}$.
2. $\boldsymbol{x} = \big(x(i,j)\big)_{i \in [n], j \in [m]}$: a **$\{0,1\}$-valued matrix** of size $n \times m$.
3. For the ***i*-th row** $\boldsymbol{x}(i,:)$, we introduce an integer r_i and impose a **constraint** on the **row sum**:
$$\mathcal{X}_{r_i} = \left\{ \boldsymbol{x}(i,:) \,\middle|\, \sum_{j \in [m]} x(i,j) = r_i \right\}.$$
4. For the ***j*-th column** $\boldsymbol{x}(:,j)$, we introduce an integer c_j and impose a **constraint** on the **column sum**:
$$\mathcal{X}_{c_j} = \left\{ \boldsymbol{x}(:,j) \,\middle|\, \sum_{i \in [n]} x(i,j) = c_j \right\}.$$

Setup

Definition

5. The set of **valid configurations** is defined to be

$$\mathcal{C} \triangleq \left\{ \boldsymbol{x} \in \{0,1\}^{n\times n} \;\middle|\; \begin{array}{c} \boldsymbol{x}(i,:) \in \mathcal{X}_{r_i},\ \forall i \in [n], \\ \boldsymbol{x}(:,j) \in \mathcal{X}_{c_j},\ \forall j \in [m] \end{array} \right\},$$

the set of **binary matrices** such that the **i-th row sum** is r_i and the **j-th column sum** is c_j.

6. We want to compute the number of the **valid configurations** $|\mathcal{C}|$.

Outline

Motivation

A setup based on binary matrices with prescribed row sums and column sums

- **Graphical-model-based approximation method**

Main results

A more general setup

Main results for a more general setup

Numerical results

Future works and connection to other works

Graphical-model-based approximation method

Main idea

1. Define a **standard factor graph (S-FG)** N whose partition function equals

$$Z(\mathsf{N}) = |\mathcal{C}|.$$

2. Run the **sum product algorithm (SPA)**, a.k.a. **belief propagation** (BP), on the S-FG N to compute the **Bethe approximation of** $|\mathcal{C}|$, denoted by $Z_{\mathrm{B}}(\mathsf{N})$.

Graphical-model-based approximation method

Example

Consider $n = m = 3$ **and** $r_i = c_j = 2$, *i.e.*, $\boldsymbol{x} \in \{0,1\}^{3\times 3}$.

The i-**th row** $\boldsymbol{x}(i,:) \in \mathcal{X}_{r_i}$ and the j-**th column** $\boldsymbol{x}(:,j) \in \mathcal{X}_{c_j}$, where

$\mathcal{X}_{r_i} = \{(1,1,0),(0,1,1),(1,0,1)\}, \quad \mathcal{X}_{c_j} = \{(1,1,0)^\mathsf{T},(0,1,1)^\mathsf{T},(1,0,1)^\mathsf{T}\}.$

1. The **local functions**:

$$f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) \triangleq \begin{cases} 1 & \text{if } \boldsymbol{x}(i,:) \in \mathcal{X}_{r_i} \\ 0 & \text{otherwise} \end{cases}, \quad f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) \triangleq \begin{cases} 1 & \text{if } \boldsymbol{x}(:,j) \in \mathcal{X}_{c_j} \\ 0 & \text{otherwise} \end{cases}.$$

2. The **support** of the local functions:

$$\mathcal{X}_{f_{\mathrm{l},i}} \triangleq \big\{\boldsymbol{x}(i,:) \in \{0,1\}^3 \;\big|\; f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) > 0\big\} = \mathcal{X}_{r_i},$$
$$\mathcal{X}_{f_{\mathrm{r},j}} \triangleq \big\{\boldsymbol{x}(:,j) \in \{0,1\}^3 \;\big|\; f_{\mathrm{r},j}\big(\boldsymbol{x}(:,j)\big) > 0\big\} = \mathcal{X}_{c_j}.$$

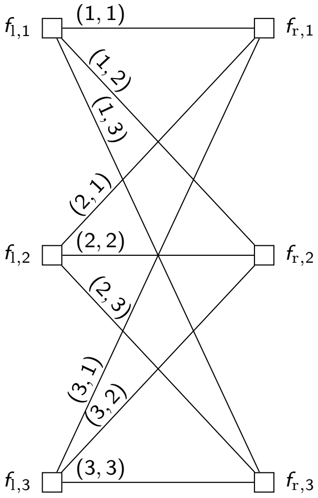
Graphical-model-based approximation method

3. The $\{0,1\}$-valued **global function**:

$$g(\boldsymbol{x}) \triangleq f_{\mathrm{l},1}\big(x(1,1), x(1,2), x(1,3)\big) \cdot f_{\mathrm{l},2}\big(x(2,1), x(2,2), x(2,3)\big) \cdots f_{\mathrm{r},2}\big(x(1,2), x(2,2), x(3,2)\big) \cdot f_{\mathrm{r},3}\big(x(1,3), x(2,3), x(3,3)\big).$$

The **previously defined** set of **valid configurations** is equal to the **support** of the global function:

$$\mathcal{C} = \left\{ \boldsymbol{x} \in \{0,1\}^{3\times 3} \;\middle|\; g(\boldsymbol{x}) > 0 \right\}.$$



4. The **partition function**:

$$Z(\mathsf{N}) \triangleq \sum_{\boldsymbol{x} \in \{0,1\}^{3\times 3}} g(\boldsymbol{x}) = |\mathcal{C}|.$$

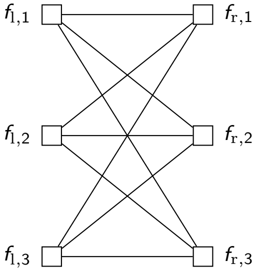
Graphical-model-based approximation method

5. The **Bethe approximation** of the partition function, *i.e.*, the **Bethe partition function**, is defined to be

$$Z_{\mathrm{B}}(\mathsf{N}) \triangleq \exp\Big(- \min_{\boldsymbol{\beta} \in \mathcal{L}(\mathsf{N})} F_{\mathrm{B}}(\boldsymbol{\beta}) \Big),$$

   where F_{B} is the **Bethe free energy (BFE)** function,

   where $\mathcal{L}(\mathsf{N})$ is the **local marginal polytope** (LMP) (see, *e.g.*,[WJ08][10]).



6. Then we run the **sum-product algorithm (SPA)**, a.k.a. **belief propagation (BP)**, on the S-FG N to get $Z_{\mathrm{B}}(\mathsf{N})$.

[10] M. J. Wainwright and M. I. Jordan, "Graphical models, exponential families, and variational inference," Foundation and Trends in Machine Learning, 2008.

Outline

Main results

1. The **projection** of the **local marginal polytope** on the **edges** in N **equals** conv($\mathcal{C}$).
   (For general S-FGs, this projection is a **relaxation** of conv($\mathcal{C}$), *i.e.*, conv($\mathcal{C}$) is a **strict subset** of this **projection**.)
2. For the **typical case** where N has an **SPA fixed point** consisting of **positive-valued messages only**,
   the SPA finds the value of $Z_{\mathrm{B}}(\mathrm{N})$ **exponentially fast**.
3. The **BFE function** has some **convexity properties**.

Comments

- A **generalization** of parts of the results in [Von13][11].
- Even though the S-FG has a **non-trivial cyclic structure**,
  the SPA has **a good performance**.

[11] P. O. Vontobel, The Bethe permanent of a nonnegative matrix," IEEE Trans. Inf. Theory, 2013.

Main results

Comments

For the setup where $n = m$, $r_i = 1$, and $c_j = 1$, it holds that

- $\mathcal{C} = \{\boldsymbol{x} \mid \boldsymbol{x}$ is a **permutation matrix** of size n-by-$n\}$
- The **projection** of the **LMP** on the **edges** equals the set of **doubly stochastic matrices** of size n-by-n.

Birkhoff–von Neumann theorem

The set of **doubly stochastic matrices** of size n-by-n is the **convex hull** of the set of the **permutation matrices** of size n-by-n.

The main result that $\mathrm{conv}(\mathcal{C})$ equals the **projection** of the **LMP** on the **edges** for our considered S-FG, can be viewed as a **generalization**.

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A more general setup

An example S-FG

Consider $n = m = 3$ and $r_i = c_j = 2$. Then

$$f_{1,i}\big(\boldsymbol{x}(i,:)\big) = \begin{cases} 1 & \text{if } \boldsymbol{x}(i,:) \in \{(1,1,0),(0,1,1),(1,0,1)\} \\ 0 & \text{otherwise} \end{cases},$$

which corresponds to a **multi-affine homogeneous real stable (MAHRS) polynomial** w.r.t. the **indeterminates** in $\boldsymbol{L} \triangleq \big(L_1, L_2, L_3\big) \in \mathbb{C}^3$:

$$p_i(\boldsymbol{L}) = \sum_{\boldsymbol{x}(i,:)\in\{0,1\}^3} f_{1,i}\big(\boldsymbol{x}(i,:)\big) \cdot \prod_{j\in[3]} \big(L_j\big)^{\boldsymbol{x}(i,j)}$$

$$= L_1 \cdot L_2 + L_2 \cdot L_3 + L_1 \cdot L_3.$$

Remark

- For details of **real stable polynomials**, see, *e.g.*,

  S. O. Gharan, "Course notes of polynomial paradigm in algorithm design," 2020, Lecture 3.

Consider a **more general** setup where **each local function** is defined based on a **(possibly different) MAHRS polynomial**.

Do the previous results **hold** in this **more general setup**?

Yes!

An MAHRS Polynomials-based S-FG

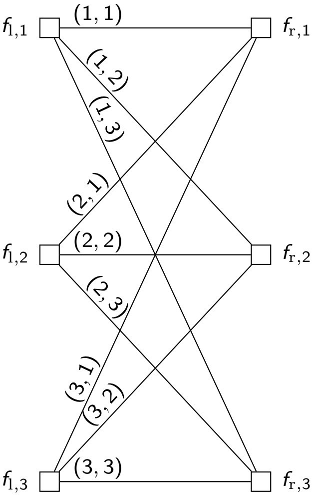
The standard factor graph (S-FG) N consists of

1. **edges**: $(1,1),(1,2),\ldots,(3,3)$;
2. **Binary** matrix

$$\boldsymbol{x} \triangleq \begin{pmatrix} x(1,1) & x(1,2) & x(1,3) \\ x(2,1) & x(2,2) & x(2,3) \\ x(3,1) & x(3,2) & x(3,3) \end{pmatrix}.$$

3. **Nonnegative-valued** local functions

   $f_{\mathrm{l},1},\ldots,f_{\mathrm{r},3}$;



An MAHRS Polynomials-based S-FG

6. The **local function** $f_{\mathrm{l},i}$ on the **LHS** is defined to be the mapping:

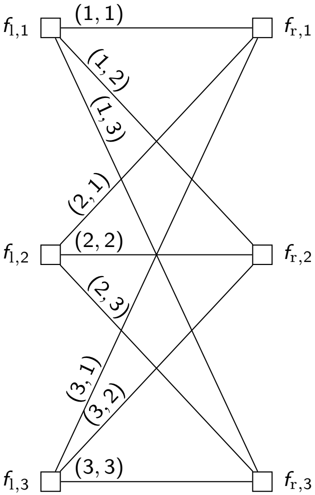
$$\{0,1\}^3 \to \mathbb{R}_{\geq 0}, \quad \boldsymbol{x}(i,:) \mapsto f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big)$$

such that it corresponds to an **MAHRS polynomial**.

7. The **support** of $f_{\mathrm{l},i}$:

$$\mathcal{X}_{f_{\mathrm{l},i}} \triangleq \left\{ \boldsymbol{x}(i,:) \in \{0,1\}^3 \;\middle|\; f_{\mathrm{l},i}\big(\boldsymbol{x}(i,:)\big) > 0 \right\}.$$

8. A **similar idea** in the definitions of $f_{\mathrm{r},j}$ and $\mathcal{X}_{f_{\mathrm{r},j}}$ on the **RHS**.



An MAHRS Polynomials-based S-FG

9. The **nonnegative-valued global function**:

$$g(\boldsymbol{x}) \triangleq f_{\mathrm{l},1}\big(\boldsymbol{x}(1,:)\big) \cdot f_{\mathrm{l},2}\big(\boldsymbol{x}(2,:)\big) \cdot f_{\mathrm{l},3}\big(\boldsymbol{x}(3,:)\big) \cdot f_{\mathrm{r},1}\big(\boldsymbol{x}(:,1)\big) \cdot f_{\mathrm{r},2}\big(\boldsymbol{x}(:,2)\big) \cdot f_{\mathrm{r},3}\big(\boldsymbol{x}(:,3)\big).$$

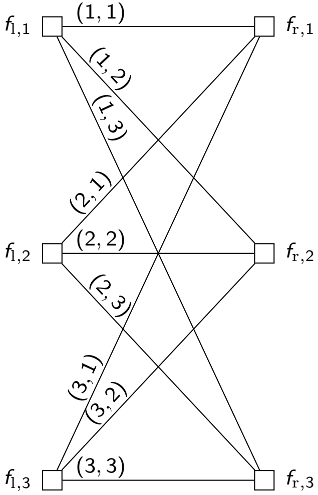
10. The set of **valid configurations**:

$$\mathcal{C} \triangleq \left\{ \boldsymbol{x} \in \{0,1\}^{3\times 3} \;\middle|\; g(\boldsymbol{x}) > 0 \right\},$$

which is also the **support** of the **global function**.

11. The **partition function**:

$$Z(\mathsf{N}) \triangleq \sum_{\boldsymbol{x}\in\mathcal{C}} g(\boldsymbol{x}).$$



Outline

Known results

Consider an **S-FG** N where **each local function** is defined based on a (possibly different) **MAHRS** polynomial.

Remarks

- Exactly computing $Z(\mathsf{N})$ is a **#P-complete problem** in general. (**#P-complete problem** is the set of the **counting problems** associated with the **decision problems** in the class **NP**.)

- **Run the SPA** to find the value of the **Bethe partition function** $Z_{\mathrm{B}}(\mathsf{N})$ that **approximates** $Z(\mathsf{N})$.

- [Theorem 3.2, SV19][12]: $Z_{\mathrm{B}}(\mathsf{N}) \leq Z(\mathsf{N})$.

[12] D. Straszak and N. K. Vishnoi, "Belief propagation, Bethe approximation, and polynomials," IEEE Trans. Inf. Theory, 2019.

Known results

Consider an **S-FG** N where **each local function** is defined based on a (possibly different) **MAHRS** polynomial.

Remarks

- **Other real-stable-polynomial-based approximation of $Z(\mathsf{N})$** [Gur15][13] and [Bra23][14].

[13] L. Gurvits, "Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: Combinatorial and algorithmic applications," Inform. and Comput., 2015.

[14] P. Brändén, J. Leake, and I. Pak, "Lower bounds for contingency tables via Lorentzian polynomials," Israel J. Math., 2023.

Main results

Consider an **S-FG** N where **each local function** is defined based on a (possibly different) **MAHRS** polynomial.

- The **support** $\mathcal{X}_{f_{\mathrm{l},i}}$ on the LHS corresponds to **a set of bases of a matroid** [Brä07][15].
- The support of the **product** of the **local functions** on the **LHS** is $\{\mathcal{X}_{f_{\mathrm{l},1}} \times \mathcal{X}_{f_{\mathrm{l},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{l},n}}\}$.
- Similarly for the local functions and the **support** on the **RHS**.
- The support of the **global function** equals the **intersection** of the bases of **matroids**:

$$\mathcal{C} = \{\mathcal{X}_{f_{\mathrm{l},1}} \times \mathcal{X}_{f_{\mathrm{l},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{l},n}}\} \bigcap \{\mathcal{X}_{f_{\mathrm{r},1}} \times \mathcal{X}_{f_{\mathrm{r},2}} \times \cdots \times \mathcal{X}_{f_{\mathrm{r},m}}\}$$

[15] P. Brändén, "Polynomials with the half-plane property and matroid theory," Adv. Math., 2007.

Main results

1. The **convex hull** $\text{conv}(\mathcal{C})$ is the **projection of the LMP** on the **edges**. (Based on results on **intersection of matroids** (see, *e.g.*, [Oxl11][16]).)

2. For the typical case where the S-FG has an SPA **fixed point** consisting of **positive-valued messages only**, the SPA finds the value of $Z_{\mathrm{B}}(\mathsf{N})$ **exponentially fast**.
   (Based on the properties of **real stable polynomials** in [Brä07][17].)

[16] J. Oxley, Matroid Theory, Oxford University Press, 2011.

[17] P. Brändén, "Polynomials with the half-plane property and matroid theory," Adv. Math., 2007.

Main results

3. The **Bethe free energy function** F_{B} has some **convexity properties**.

The proof of the convexity is **new**.

This result is based on the **dual** form of $Z_{\mathrm{B}}(\mathsf{N})$ in the following two papers:

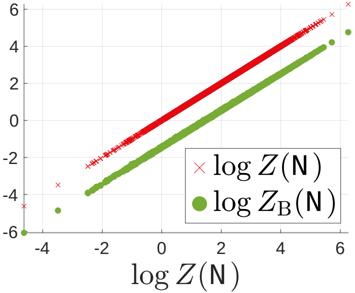
- D. Straszak and N. K. Vishnoi, "Belief propagation, Bethe approximation, and polynomials," IEEE Trans. Inf. Theory, 2019.
- N. Anari and S. O. Gharan, "A generalization of permanent inequalities and applications in counting and optimization," Adv. Math., 2021.

Outline

Numerical results

Setup

- We first consider the case $n = m = 6$ **and** $r_i = c_j = 2$.
- We independently randomly generate 3000 **instances** of N.



Observation

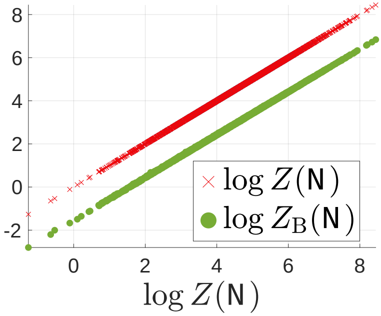
- $Z_{\mathrm{B}}(\mathsf{N}) \leq Z(\mathsf{N})$ ([Theorem 3.2, SV19][18]).
- $Z_{\mathrm{B}}(\mathsf{N})$ provides a **good estimate** of $Z(\mathsf{N})$ in this case.

[18] D. Straszak and N. K. Vishnoi, "Belief propagation, Bethe approximation, and polynomials," IEEE Trans. Inf. Theory, 2019.

Numerical results

Setup

Consider **the same setup** as the previous case, but with $n = m = 6$ **replaced** by $n = m = 7$.



Observation

We can make **similar observations**.

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Numerical results

- **Future works and connection to other works**

Future work

- Consider a **more general** S-FG, where each local function corresponds to a **more general** polynomial.
- Prove the **convergence** of the SPA for a **more general** S-FG.

Connection to other works

Works on **polynomial approaches** to approximate **partition functions**:

- L. Gurvits, "Unleashing the power of Schrijver's permanental inequality with the help of the Bethe approximation," Elec. Coll. Comp. Compl., 2011.
- D. Straszak and N. K. Vishnoi, "Belief propagation, Bethe approximation, and polynomials," IEEE Trans. Inf. Theory, 2019.
- N. Anari and S. O. Gharan, "A generalization of permanent inequalities and applications in counting and optimization," Adv. Math., 2021.

Connection to other works

Works on the properties of **real stable** polynomials and the **partition functions**.

- P. Brändén, "The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability," Amer. J. Math., 2014.
- J. Borcea and P. Brändén, "The Lee-Yang and Pólya-Schur programs. II. Theory of stable polynomials and applications," Commun. Pure Appl. Math., 2009.
- J. Borcea, P. Brändén, and T. M. Liggett, "Negative dependence and the geometry of polynomials," J. Amer. Math. Soc., 2009.

Thank you!

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